Gravitational Core of Double Field Theory

—Lecture Notes—

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Abstract

Double Field Theory (DFT) has emerged as a comprehensive framework for gravity, presenting a testable and robust alternative to General Relativity (GR), rooted in the $\mathbf{O}(D,D)$ symmetry principle of string theory. These lecture notes aim to provide an accessible introduction to DFT, structured in a manner similar to traditional GR courses. Key topics include doubled-yet-gauged coordinates, Riemannian *versus* non-Riemannian parametrisations of fundamental fields, covariant derivatives, curvatures, and the $\mathbf{O}(D,D)$-symmetric augmentation of the Einstein field equation, identified as the unified field equation for the closed string massless sector. By offering a novel perspective, DFT addresses unresolved questions in GR and enables the exploration of diverse physical phenomena, paving the way for significant future research.

Preface: A Physicist's Apology for Another Review[1]

Double Field Theory (DFT) was initially conceived by Siegel in 1993 [2, 3] and independently developed, along with its name, by Hull and Zwiebach in 2009 [4, 5], with further refinements introduced in 2010 [6, 7]. The term "double" reflects the use of doubled coordinates, $x^A = (\tilde{x}_\mu, x^\nu)$, a concept originally introduced by Duff in 1990 [8], to describe D-dimensional physics, in contrast to the conventional use of x^μ alone.

Three influential review papers on DFT [9–11], all written in 2013, provided timely insights into the field during its early development within the framework of string theory. Over the past decade, significant progress has been continuously made, including advancements, in particular, in cosmological, black hole-related, and phenomenological aspects of DFT *e.g.,* [12–24], with more developments to be discussed later.

Furthermore, a unified field equation for DFT was proposed in [25], co-authored by the present author, which generalises the Einstein field equation of General Relativity by incorporating $\mathbf{O}(D, D)$ symmetry. Through subsequent developments, DFT has, at least in the author's view, matured into a self-sustaining framework for gravity, characterised by its predictive and falsifiable nature.

These lecture notes are intended to provide an accessible introduction to the foundations of DFT, structured in a manner analogous to introductory General Relativity courses. The aim is to engage not only the `hep-th` community but also the `gr-qc` community in exploring and testing this remarkable and precise modified theory of gravity, as predicted by string theory.

These notes were developed in response to repeated requests from colleagues and participants at recent workshops held in 2025 [26, 27]. They are based on the author's lectures from the HANGANG Gravity Workshop in 2024 and a graduate-level course at Sogang University in 2025.

Yet, this modest contribution traces its origins to the author's first encounter with DFT during Zwiebach's lecture at a summer school in München in 2010 [28]. That lecture sparked the author's journey into the subject, which began in November 2010 with an exploration of the DFT analogue of the Christoffel symbols in collaboration with Imtak Jeon and Kanghoon Lee [29].

Although most $\mathbf{O}(D, D)$-manifest computations in DFT can be performed with pen and paper, deriving explicit expressions for $\mathbf{O}(D, D)$-symmetric covariant derivatives and curvatures in terms of their $\mathbf{O}(D, D)$-breaking component fields, such as $\{g_{\mu\nu}, B_{\mu\nu}, \phi\}$, often requires computational tools like *Cadabra* [30].

[1] *c.f.* A Mathematician's Apology by G. H. Hardy [1].

Contents

1 Introduction

In electrodynamics, the electric field is conventionally represented by the capital letter $\mathbf{E}$ for obvious reasons. However, the magnetic field is represented by either $\mathbf{B}$ or $\mathbf{H}$ rather than $\mathbf{M}$. This convention arises because Maxwell did not use vector notations when formulating his equations in 1861. Instead, Maxwell used the letters $\{E, F, G\}$ to denote components of the electric field and neighboring letters, such as $\{B, C, D\}$ or $\{H, I, J\}$, for the magnetic field. It was Heaviside—or perhaps the rotational $\mathbf{SO}(3)$ symmetry—that reformulated Maxwell's equations into their modern vectorial form:

$$\nabla \cdot \mathbf{E} = \rho\,, \qquad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}\,, \qquad \nabla \cdot \mathbf{B} = 0\,, \qquad \nabla \times \mathbf{B} = \mathbf{J} + \frac{\partial \mathbf{E}}{\partial t}\,. \tag{1.1}$$

Minkowski, or the $\mathbf{SO}(1,3)$ Lorentz symmetry, then introduced further simplifications in 1908, unifying space and time into a four-dimensional spacetime framework and recasting Maxwell's equations in a more compact and elegant form:

$$\partial_\lambda F^{\lambda\mu} = J^\mu\,, \qquad \partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = 0\,. \tag{1.2}$$

In string theory, a similar unification has been accomplished through the $\mathbf{O}(D,D)$ symmetry principle. The vanishing of the three beta-functions on a closed string worldsheet,

$$\begin{aligned} R_{\mu\nu} + 2\bigtriangledown_\mu(\partial_\nu\phi) - \tfrac{1}{4}H_{\mu\rho\sigma}H_\nu{}^{\rho\sigma} &= 0\,, \\ \tfrac{1}{2}e^{2\phi}\bigtriangledown^\rho\big(e^{-2\phi}H_{\rho\mu\nu}\big) &= 0\,, \\ R + 4\Box\phi - 4\partial_\mu\phi\partial^\mu\phi - \tfrac{1}{12}H_{\lambda\mu\nu}H^{\lambda\mu\nu} &= 0\,, \end{aligned} \tag{1.3}$$

is unified into a single formula characterized by the vanishing of the $\mathbf{O}(D,D)$-symmetric augmentation of the Einstein curvature tensor [31],

$$G_{AB} = 0\,, \tag{1.4}$$

which corresponds to the vacuum case of the more general unified field equation in Double Field Theory (DFT), dubbed the *Einstein Double Field Equation* (EDFE) [25]:

$$G_{AB} = T_{AB}\,. \tag{1.5}$$

Hereafter, the capital letters $A, B, \cdots$ denote $\mathbf{O}(D,D)$ vector indices, which span the doubled spacetime dimensions, $D+D$. As reviewed below, the DFT Einstein curvature, G_{AB}, on the left-hand side of the equation, can be—but not necessarily—constructed from the trio $\{g_{\mu\nu}, B_{\mu\nu}, \phi\}$. This trio represents the traditional closed string massless sector, known as the Neveu–Schwarz Neveu–Schwarz (NSNS) sector, and constitutes the stringy gravitational degrees of freedom. On the right-hand side, the DFT energy-momentum tensor, T_{AB}, accounts for other sectors (or "matter"), including the Ramond–Ramond (RR) sector and NSR fermions. Like General Relativity, these quantities are governed by conservation laws derived from the action principle:

$$\nabla_A G^A{}_B = 0 \ : \ \text{off-shell}\,, \qquad\qquad \nabla_A T^A{}_B = 0 \ : \ \text{on-shell}\,. \tag{1.6}$$

This conservation principle highlights the fundamental symmetries and consistency of the theory, mirroring General Relativity (GR). With a greater number of components in the energy-momentum tensor, the gravitational physics in DFT becomes inherently richer compared to GR.

Summary. In analogy with Lorentz symmetry unifying Maxwell's equations, $\mathbf{O}(D, D)$ symmetry unifies the closed string beta-function equations. The EDFE (1.5) is a generalised and testable extension of Einstein's field equation, providing a richer framework for gravity.

2 Double Field Theory Minimum

In this main section, we delve into the geometric foundations of Double Field Theory (DFT). After establishing the necessary notation, we introduce the concept of double-yet-gauged coordinates and the fundamental field variables, followed by the formulation of semi-covariant derivatives and curvatures. The notion of 'semi-covariance' acts as an intermediate step in the formalism, leading to fully covariant derivatives and curvatures. The overarching goal is to present the action principle and derive the unified field equation, the EDFE (1.5), along with the conservation law (1.6).

2.1 Symmetry & Notation

Aim. To establish index conventions and identify the symmetry group $\mathbf{O}(D, D)$ as well as the twofold Lorentz symmetries.

DFT can be viewed as a top-down extension of GR, guided by the $\mathbf{O}(D, D)$ symmetry principle, necessitating the introduction of notation for its fundamental symmetry groups: $\mathbf{O}(D, D)$ and $\mathbf{Spin}(1, D-1) \times \mathbf{Spin}(D-1, 1)$. As summarised in Table 1, capital letters are used for $\mathbf{O}(D, D)$ vector indices, while small letters are reserved for the twofold local Lorentz symmetries. To distinguish between the indices of the two distinct spin groups, barred and unbarred notations are employed: unbarred indices correspond to $\mathbf{Spin}(1,D-1)$, while barred indices correspond to $\mathbf{Spin}(D-1,1)$. These conventions prepare the ground for consistently formulating doubled geometry.

Symmetry	Index	"Metric" (raising/lowering indices)
$\mathbf{O}(D,D)$	$A,B,\cdots,M,N,\cdots$	$\mathcal{J}_{AB}=\left(\begin{array}{cc}0&1\\1&0\end{array}\right)$
$\mathbf{Spin}(1,D{-}1)$	$p,q,\cdots$	$\eta_{pq}=\mathrm{diag}(-++\cdots+)$
$\mathbf{Spin}(D{-}1,1)$	$\bar{p},\bar{q},\cdots$	$\bar{\eta}_{\bar{p}\bar{q}}=\mathrm{diag}(+--\cdots-)$

Table 1: Vectorial indices and "metrics" for $\mathbf{O}(D,D)$, $\mathbf{Spin}(1,D-1)$, and $\mathbf{Spin}(D-1,1)$:
i) $\mathcal{J}_{AB}$ denotes the invariant metric for the $\mathbf{O}(D,D)$ group.
ii) η_{pq} and $\bar{\eta}_{\bar{p}\bar{q}}$ are flat D-dimensional Minkowskian metrics associated with $\mathbf{Spin}(1,D-1)$ and $\mathbf{Spin}(D-1,1)$, respectively. These metrics exhibit opposite signatures.

2.2 Doubled-yet-Gauged Coordinates & Diffeomorphisms

Aim. To introduce doubled coordinates, the section condition, and relevant diffeomorphisms.

The $\mathbf{O}(D,D)$-invariant metric $\mathcal{J}_{AB}$ presented in Table 1 plays a crucial role in splitting the doubled coordinates into two parts, $x^A=(\tilde{x}_\mu,x^\nu)$, where $\mu,\nu=0,1,2,\cdots,D{-}1$ denote D-dimensional spacetime indices. A fundamental requirement in DFT is that all functions in the theory, collectively denoted as $\{\Phi,\hat{\Phi},\tilde{\Phi},\cdots\}$—including physical fields and local gauge parameters—must satisfy the *section condition* [36]. This condition restricts their dependence on the doubled coordinates via the constraint:

$$\partial_A\partial^A=\partial_\mu\tilde{\partial}^\mu+\tilde{\partial}^\mu\partial_\mu=0\,,\tag{2.1}$$

which ensures that any pairwise contraction of (doubled) partial derivatives vanishes:

$$\partial_A\partial^A\Phi=0\,,\qquad\qquad \partial_A\Phi\partial^A\hat{\Phi}=0\,,\qquad\qquad \partial_A\partial^A(\Phi\hat{\Phi})=0\,.\tag{2.2}$$

In practice, the section condition is often solved by setting $\tilde{\partial}^{\mu} \equiv 0$, thereby eliminating dependence on the dual coordinates $\tilde{x}^{\mu}$. The general solution, or "section", is obtained by applying a global $\mathbf{O}(D,D)$ rotation to this specific choice. The $\mathbf{O}(D,D)$ symmetry is spontaneously broken by selecting the D-dimensional section, such as $\tilde{\partial}^{\mu} \equiv 0$. However, in cases where a configuration admits an isometric direction, *e.g.* $\partial_1 \equiv 0$, the $\mathbf{O}(1,1)$ rotation of the $(\tilde{x}_1, x^1)$ plane generates new configurations, a procedure known as Buscher's rule [37, 38].

The section condition is equivalent to a certain translational invariance generated by *derivative-index-valued* shift parameters Δ^A [39, 40]:

$$\Phi(x) = \Phi(x+\Delta) \qquad \text{where} \qquad \Delta^A = \hat{\Phi}\partial^A\tilde{\Phi}\,, \qquad \Delta^A\partial_A = 0\,. \tag{2.3}$$

The geometric meaning of the section condition, as suggested in [39], is that the doubled coordinates are *gauged* by the derivative-index-valued vectors:

$$x^A \ \sim\ x^A + \Delta^A\,, \qquad\qquad \Delta^A\partial_A = 0\,. \tag{2.4}$$

In this framework, each gauge orbit corresponds to a single spacetime point. The concept of "derivative-index-valued" vectors is made possible by the $\mathbf{O}(D,D)$-invariant metric $\mathcal{J}_{AB}$, which raises the vector index of the partial derivative: $\partial^A = \mathcal{J}^{AB}\partial_B$.

In DFT, diffeomorphisms are governed by a generalised Lie derivative, denoted as $\hat{\mathcal{L}}_{\xi}$ [3, 5]:

$$\hat{\mathcal{L}}_{\xi}T_{A_1\cdots A_n} := \xi^B\partial_B T_{A_1\cdots A_n} + \omega\partial_B\xi^B T_{A_1\cdots A_n} + \sum_{i=1}^{n}(\partial_{A_i}\xi_B - \partial_B\xi_{A_i})T_{A_1\cdots A_{i-1}}{}^{B}{}_{A_{i+1}\cdots A_n}\,, \tag{2.5}$$

where ω is the weight of the tensor or tensor density, and each tensor index A_i is rotated by an infinitesimal $\mathbf{so}(D,D)$ element, $\partial_{A_i}\xi_B - \partial_B\xi_{A_i}$. For consistency, generalised Lie derivatives are closed under commutation, provided the section condition holds:

$$\left[\hat{\mathcal{L}}_{\zeta}, \hat{\mathcal{L}}_{\xi}\right] = \hat{\mathcal{L}}_{[\zeta,\xi]_{\mathrm{C}}}\,, \tag{2.6}$$

where the commutator is given by the C-bracket:

$$[\zeta,\xi]^A_{\mathrm{C}} = \tfrac{1}{2}\left(\hat{\mathcal{L}}_{\zeta}\xi^A - \hat{\mathcal{L}}_{\xi}\zeta^A\right) = \zeta^B\partial_B\xi^A - \xi^B\partial_B\zeta^A + \tfrac{1}{2}\xi^B\partial^A\zeta_B - \tfrac{1}{2}\zeta^B\partial^A\xi_B\,. \tag{2.7}$$

It is noteworthy that the $\mathbf{O}(D,D)$-metric is invariant under generalised diffeomorphisms, $\hat{\mathcal{L}}_{\xi}\mathcal{J}_{AB} = 0$, and the generalised Lie derivative itself is covariant [25]:

$$\delta_{\xi}(\hat{\mathcal{L}}_{\zeta}T_{A_1\cdots A_n}) = \hat{\mathcal{L}}_{\zeta}(\delta_{\xi}T_{A_1\cdots A_n}) + \hat{\mathcal{L}}_{\delta_{\xi}\zeta}T_{A_1\cdots A_n} = \hat{\mathcal{L}}_{\xi}(\hat{\mathcal{L}}_{\zeta}T_{A_1\cdots A_n})\,. \tag{2.8}$$

✓ Explicit Gauging of the Doubled Coordinates

The section condition encapsulates the concept of *doubled-yet-gauged coordinates* [39] in DFT, where coordinates serve only as labels for the dynamical fields. This contrasts with the particle worldline or string worldsheet actions, where the target spacetime coordinates themselves become dynamical fields and must be explicitly gauged [40–43].

The usual coordinate basis of differential one-forms, $\mathrm{d}x^A$, is not $\mathbf{O}(D,D)$-symmetric under infinitesimal passive coordinate transformations,[2] $x^A \to x^A + \xi^A$, as shown below:

$$\delta(\mathrm{d}x^A) = \mathrm{d}\xi^A = \mathrm{d}x^B\partial_B\xi^A \neq \mathrm{d}x^B(\partial_B\xi^A - \partial^A\xi_B)\,. \tag{2.9}$$

To restore $\mathbf{O}(D,D)$ symmetry, a derivative-index-valued gauge potential is introduced to explicitly gauge the doubled coordinates. This defines a new gauged differential:

$$\mathrm{D}x^A := \mathrm{d}x^A - \mathfrak{a}^A\,, \qquad\qquad \mathfrak{a}^A\partial_A = 0\,. \tag{2.10}$$

The passive transformations for the coordinates x^A and the gauge potential $\mathfrak{a}^A$ are given by:

$$\delta x^A = \xi^A\,, \qquad \delta\mathfrak{a}^A = \mathrm{D}x^B\partial^A\xi_B\,, \qquad \delta\mathfrak{a}^A\partial_A = 0\,. \tag{2.11}$$

This leads to an $\mathbf{O}(D,D)$-symmetric transformation for the gauged differential:

$$\delta(\mathrm{D}x^A) = \mathrm{D}x^B(\partial_B\xi^A - \partial^A\xi_B)\,. \tag{2.12}$$

To measure the distance between gauge orbits, a proper length can be defined through a path integral [44]:

$$L = -\ln\left[\int\mathfrak{D}\mathfrak{a}\,\exp\left(-\int\sqrt{\mathrm{D}x^A\mathrm{D}x^B\mathcal{H}_{AB}}\right)\right]\,, \tag{2.13}$$

where $\mathrm{D}x^A$ is the gauged coordinate differential defined in (2.10), $\mathfrak{a}$ is the auxiliary gauge potential to be integrated out, and $\mathcal{H}_{AB}$ is the DFT-metric, or the generalised metric, which will be detailed later.

This definition of proper length naturally leads to a doubled worldline action for a particle [42]:

$$S = \frac{1}{l_s}\int\mathrm{d}\tau\,\frac{1}{2}e^{-1}\,\mathrm{D}_\tau x^A\mathrm{D}_\tau x^B\mathcal{H}_{AB}(x) - \frac{1}{2}(m\,l_s)^2e\,, \tag{2.14}$$

[2] Active transformations are set by the generalised Lie derivative (2.5).

which further generalises to a completely covariant, doubled worldsheet action for a string [40, 45],

$$S = \frac{1}{4\pi\alpha'}\int \mathrm{d}^2\sigma \ -\tfrac{1}{2}\sqrt{-h}h^{\alpha\beta}\mathrm{D}_\alpha x^M \mathrm{D}_\beta x^N \mathcal{H}_{MN} - \epsilon^{\alpha\beta}\mathrm{D}_\alpha x^M \mathfrak{a}_{\beta M}\,. \tag{2.15}$$

This framework can be extended to a κ-symmetric Green–Schwarz superstring, as discussed in [41].

Summary. The doubled-yet-gauged construction preserves $\mathbf{O}(D,D)$ symmetry while constraining dynamics to consistent D-dimensional sections. The gauging procedure leads to $\mathbf{O}(D,D)$-symmetric particle and string actions, extendable to supersymmetric cases.

2.3 Fundamental Field Variables: Non-Riemannian Geometry

Aim. To define the generalised metric and dilaton, introduce projection operators and vielbeins, and set up the non-Riemannian sector.

The fundamental fields of DFT are the generalised metric, $\mathcal{H}_{AB}$, and the dilaton, d, which together define the geometric and gravitational structure of the theory. These fields encapsulate the stringy nature of spacetime geometry and play a central role in the formulation of DFT.

By definition, the generalised metric $\mathcal{H}_{AB}$ is a symmetric $\mathbf{O}(D,D)$ element:

$$\mathcal{H}_{AB} = \mathcal{H}_{BA}\,, \qquad\qquad \mathcal{H}_A{}^C\mathcal{H}_B{}^D\mathcal{J}_{CD} = \mathcal{J}_{AB}\,, \tag{2.16}$$

and the exponentiated form of the dilaton, e^{-2d}, serves as the integral measure in DFT: it is a scalar density of unit weight. In words, the generalised metric squares to the identity: $\mathcal{H}_A{}^B\mathcal{H}_B{}^C = \delta_A{}^C$. Their generalised Lie derivatives (2.5) are explicitly given by

$$\hat{\mathcal{L}}_\xi\mathcal{H}_{AB} = \xi^C\partial_C\mathcal{H}_{AB} + 2\partial_{[A}\xi_{C]}\mathcal{H}^C{}_B + 2\partial_{[B}\xi_{C]}\mathcal{H}_A{}^C\,, \qquad \hat{\mathcal{L}}_\xi d = -\tfrac{1}{2}e^{2d}\hat{\mathcal{L}}_\xi\left(e^{-2d}\right) = \xi^A\partial_A d - \tfrac{1}{2}\partial_A\xi^A\,. \tag{2.17}$$

Combining the invariant metric $\mathcal{J}_{AB}$ and the generalised metric $\mathcal{H}_{AB}$, we define symmetric projection matrices [29]:

$$\begin{aligned} P_{AB} &= P_{BA} = \tfrac{1}{2}(\mathcal{J}_{AB} + \mathcal{H}_{AB})\,, \qquad & P_A{}^B P_B{}^C &= P_A{}^C\,, \\ \bar{P}_{AB} &= \bar{P}_{BA} = \tfrac{1}{2}(\mathcal{J}_{AB} - \mathcal{H}_{AB})\,, \qquad & \bar{P}_A{}^B \bar{P}_B{}^C &= \bar{P}_A{}^C\,, \end{aligned} \tag{2.18}$$

which are orthogonal and complete:

$$P_A{}^B\bar{P}_B{}^C = 0\,, \qquad\qquad P_A{}^B + \bar{P}_A{}^B = \delta_A{}^B\,. \tag{2.19}$$

Taking the "square roots" of the projection matrices, we introduce a pair of vielbeins, V_{Ap} and $\bar{V}_{B\bar{q}}$:

$$P_{AB} = V_A{}^p V_B{}^q \eta_{pq}\,, \qquad\qquad \bar{P}_{AB} = \bar{V}_A{}^{\bar{p}} \bar{V}_B{}^{\bar{q}} \bar{\eta}_{\bar{p}\bar{q}}\,. \tag{2.20}$$

The vielbeins satisfy the following defining property:

$$V_A{}^p V_B{}^q \eta_{pq} + \bar{V}_A{}^{\bar{p}} \bar{V}_B{}^{\bar{q}} \bar{\eta}_{\bar{p}\bar{q}} = \mathcal{J}_{AB}\,. \tag{2.21}$$

Essentially, viewed as a $(D+D)\times(D+D)$ matrix, the pair $\big(V_A{}^p, \bar{V}_A{}^{\bar{p}}\big)$ simultaneously diagonalises $\mathcal{J}_{AB}$ and $\mathcal{H}_{AB}$ into diag$(\eta, +\bar{\eta})$ and diag$(\eta, -\bar{\eta})$, respectively. Furthermore, since the left inverse coincides with the right inverse, this property (2.21) is equivalent to:

$$V_{Ap} V^A{}_q = \eta_{pq}\,, \qquad \bar{V}_{A\bar{p}} \bar{V}^A{}_{\bar{q}} = \bar{\eta}_{\bar{p}\bar{q}}\,, \qquad V_{Ap} \bar{V}^A{}_{\bar{q}} = 0\,. \tag{2.22}$$

It follows that

$$P_A{}^B V_{Bp} = V_{Ap}\,, \qquad \bar{P}_A{}^B \bar{V}_{B\bar{p}} = \bar{V}_{A\bar{p}}\,, \qquad P_A{}^B \bar{V}_{B\bar{p}} = 0 = \bar{P}_A{}^B V_{Bp}\,. \tag{2.23}$$

The presence of twofold vielbeins, $V_A{}^p$ and $\bar{V}_A{}^{\bar{p}}$, and spin groups, $\mathbf{Spin}(1, D{-}1) \times \mathbf{Spin}(D{-}1, 1)$, is a distinctly string-theoretic feature. This reflects the existence of two separate locally inertial frames: one associated with the left-moving sector, and the other associated with the right-moving sector of closed strings [32]. Naturally, there are also two types of spinorial fermions, as seen in supersymmetric DFTs [33, 34] and as conjectured in the Standard Model of particle physics coupled to DFT [35]. This dual structure could offer testable predictions for DFT and string theory. Moreover, it facilitates the unification of type IIA and IIB supergravities in the $D = 10$ maximally supersymmetric DFT [34] that has been explicitly constructed to the complete, quartic order in fermions.

The generalised metric and the vielbeins are constrained by the properties (2.16), (2.21), and (2.22). Consequently, their infinitesimal variations satisfy the following relations:[3]

$$\delta\mathcal{H}_{AB} = (P\delta\mathcal{H}\bar{P})_{AB} + (\bar{P}\delta\mathcal{H}P)_{AB}\,, \tag{2.24}$$

and

$$\delta V_{Ap} = \bar{V}_{A\bar{q}} \bar{V}^{B\bar{q}} \delta V_{Bp} + V_A{}^q V_{B[q} \delta V^B{}_{p]}\,, \qquad \delta\bar{V}_{A\bar{p}} = V_{Aq} V^{Bq} \delta\bar{V}_{B\bar{p}} + \bar{V}_A{}^{\bar{q}} \bar{V}_{B[\bar{q}} \delta\bar{V}^B{}_{\bar{p}]}\,, \tag{2.25}$$

where the second terms on the right-hand sides correspond to local Lorentz rotations.

[3] Observe the implicit index contractions in $(P\delta\mathcal{H}\bar{P})_{AB}$, where the expression expands as $P_A{}^C \delta\mathcal{H}_{CD} \bar{P}^D{}_B$.

✓ **Parametrisations: Riemannian *vs.* non-Riemannian**

> **Aim.** To present explicit parametrisations of the generalised metric in both Riemannian and non-Riemannian forms.

DFT and its supersymmetric extensions offer a unified framework to describe stringy spacetime dynamics via the fundamental fields, $\{\mathcal{H}_{AB}, d\}$ and $\{V_{Ap}, \bar{V}_{B\bar{q}}, d\}$. These fields satisfy key defining relations, such as (2.16) and (2.21), and allow for the classification of DFT geometries through the introduction of two non-negative integers, $(n, \bar{n})$, whose sum is bounded: $0 \leq n + \bar{n} \leq D$.

With $1 \leq i, j \leq n$ and $1 \leq \bar{\imath}, \bar{\jmath} \leq \bar{n}$, the DFT metric satisfying (2.16) takes the most general form [44],

$$\mathcal{H}_{AB} = \left(\begin{array}{cc} \mathcal{H}^{\mu\nu} & -\mathcal{H}^{\mu\sigma}B_{\sigma\lambda} + Y^{\mu}_{i}X^{i}_{\lambda} - \bar{Y}^{\mu}_{\bar{\imath}}\bar{X}^{\bar{\imath}}_{\lambda} \\ B_{\kappa\rho}\mathcal{H}^{\rho\nu} + X^{i}_{\kappa}Y^{\nu}_{i} - \bar{X}^{\bar{\imath}}_{\kappa}\bar{Y}^{\nu}_{\bar{\imath}} & \mathcal{K}_{\kappa\lambda} - B_{\kappa\rho}\mathcal{H}^{\rho\sigma}B_{\sigma\lambda} + 2X^{i}_{(\kappa}B_{\lambda)\rho}Y^{\rho}_{i} - 2\bar{X}^{\bar{\imath}}_{(\kappa}B_{\lambda)\rho}\bar{Y}^{\rho}_{\bar{\imath}} \end{array} \right) . \tag{2.26}$$

Here, $\mathcal{H}$ and $\mathcal{K}$ are symmetric, B is skew-symmetric,

$$\mathcal{H}^{\mu\nu} = \mathcal{H}^{\nu\mu}\,, \qquad \mathcal{K}_{\mu\nu} = \mathcal{K}_{\nu\mu}\,, \qquad B_{\mu\nu} = -B_{\nu\mu}\,; \tag{2.27}$$

$\mathcal{H}$ and $\mathcal{K}$ admit kernels,

$$\mathcal{H}^{\mu\nu}X^{i}_{\nu} = 0 = \mathcal{H}^{\mu\nu}\bar{X}^{\bar{\imath}}_{\nu}\,, \qquad \mathcal{K}_{\mu\nu}Y^{\nu}_{j} = 0 = \mathcal{K}_{\mu\nu}\bar{Y}^{\nu}_{\bar{\jmath}}\,; \tag{2.28}$$

and a completeness relation must hold,

$$\mathcal{H}^{\mu\rho}\mathcal{K}_{\rho\nu} + Y^{\mu}_{i}X^{i}_{\nu} + \bar{Y}^{\mu}_{\bar{\imath}}\bar{X}^{\bar{\imath}}_{\nu} = \delta^{\mu}{}_{\nu}\,. \tag{2.29}$$

The linear independence of the kernel's zero-eigenvectors implies that

$$Y^{\mu}_{i}X^{j}_{\mu} = \delta_{i}{}^{j}\,, \quad \bar{Y}^{\mu}_{\bar{\imath}}\bar{X}^{\bar{\jmath}}_{\mu} = \delta_{\bar{\imath}}{}^{\bar{\jmath}}\,, \quad Y^{\mu}_{i}\bar{X}^{\bar{\jmath}}_{\mu} = 0 = \bar{Y}^{\mu}_{\bar{\imath}}X^{j}_{\mu}\,, \quad \mathcal{H}^{\rho\mu}\mathcal{K}_{\mu\nu}\mathcal{H}^{\nu\sigma} = \mathcal{H}^{\rho\sigma}\,, \quad \mathcal{K}_{\rho\mu}\mathcal{H}^{\mu\nu}\mathcal{K}_{\nu\sigma} = \mathcal{K}_{\rho\sigma}\,, \tag{2.30}$$

and hence the $\mathbf{O}(D,D)$-invariant trace of the DFT metric amounts to $\mathcal{H}_{A}{}^{A} = 2(n - \bar{n})$.

The precise expression of the $(n, \bar{n})$ DFT metric (2.26) and the fundamental algebraic relations (2.28), (2.29) are all invariant under $\mathbf{GL}(n) \times \mathbf{GL}(\bar{n})$ local rotations,

$$\left(X^{i}_{\mu}\,,\, Y^{\mu}_{i}\,,\, \bar{X}^{\bar{\imath}}_{\mu}\,,\, \bar{Y}^{\nu}_{\bar{\imath}}\right) \quad \longmapsto \quad \left(X^{j}_{\mu}\,R_{j}{}^{i}\,,\, R^{-1}{}_{i}{}^{j}\,Y^{\nu}_{j}\,,\, \bar{X}^{\bar{\jmath}}_{\mu}\,\bar{R}_{\bar{\jmath}}{}^{\bar{\imath}}\,,\, \bar{R}^{-1}{}_{\bar{\imath}}{}^{\bar{\jmath}}\,\bar{Y}^{\nu}_{\bar{\jmath}}\right)\,, \tag{2.31}$$

and, with arbitrary local parameters, $V_{\mu i}, \bar{V}_{\mu\bar{\imath}}$, under the following transformations,

$$
\begin{array}{lll}
Y_i^\mu & \longmapsto & Y_i^\mu + \mathcal{H}^{\mu\nu}V_{\nu i}\,, \\
\bar{Y}_{\bar{\imath}}^\mu & \longmapsto & \bar{Y}_{\bar{\imath}}^\mu + \mathcal{H}^{\mu\nu}\bar{V}_{\nu\bar{\imath}}\,, \\
\mathcal{K}_{\mu\nu} & \longmapsto & \mathcal{K}_{\mu\nu} - 2X^i_{(\mu}\mathcal{K}_{\nu)\rho}\mathcal{H}^{\rho\sigma}V_{\sigma i} - 2\bar{X}^{\bar{\imath}}_{(\mu}\mathcal{K}_{\nu)\rho}\mathcal{H}^{\rho\sigma}\bar{V}_{\sigma\bar{\imath}} + (X^i_\mu V_{\rho i} + \bar{X}^{\bar{\imath}}_\mu \bar{V}_{\rho\bar{\imath}})\mathcal{H}^{\rho\sigma}(X^j_\nu V_{\sigma j} + \bar{X}^{\bar{\jmath}}_\nu \bar{V}_{\sigma\bar{\jmath}})\,, \\
B_{\mu\nu} & \longmapsto & B_{\mu\nu} - 2X^i_{[\mu}V_{\nu]i} + 2\bar{X}^{\bar{\imath}}_{[\mu}\bar{V}_{\nu]\bar{\imath}} + 2X^i_{[\mu}\bar{X}^{\bar{\imath}}_{\nu]}\left(Y^\rho_i\bar{V}_{\rho\bar{\imath}} + \bar{Y}^\rho_{\bar{\imath}}V_{\rho i} + V_{\rho i}\mathcal{H}^{\rho\sigma}\bar{V}_{\sigma\bar{\imath}}\right)\,.
\end{array}
\tag{2.32}
$$

In fact, these two symmetries (2.31), (2.32) are identifiable as the twofold local Lorentz symmetries.

On the one hand, in the case $(n,\bar{n}) = (0,0)$, $\mathcal{K}_{\mu\nu}$ and $\mathcal{H}^{\mu\nu}$ coincide with the usual (invertible) Riemannian metric and its inverse. In this Riemannian limit, the $(0,0)$ DFT metric takes the familiar form [46],

$$
\mathcal{H}_{AB} = \left(\begin{array}{cc} g^{\mu\nu} & -g^{\mu\lambda}B_{\lambda\tau} \\ B_{\sigma\kappa}g^{\kappa\nu} & g_{\sigma\tau} - B_{\sigma\kappa}g^{\kappa\lambda}B_{\lambda\tau} \end{array}\right)\,. \tag{2.33}
$$

The $\mathbf{O}(D,D)$-symmetric proper length (2.13), as well as the doubled particle and string actions (2.14), (2.15), all reduce to their conventional (undoubled) forms after integrating out the auxiliary variables $\mathfrak{a}^A$.

On the other hand, cases with $(n,\bar{n}) \neq (0,0)$ are inherently non-Riemannian, as they lack an invertible Riemannian metric. Specifically, in the cases $(n,\bar{n}) = (D,0)$ or $(0,D)$, $\mathcal{K}_{\mu\nu}$ and $\mathcal{H}^{\mu\nu}$ vanish, the relations $Y_i^\mu X_\nu^i = \delta^\mu{}_\nu$ or $\bar{Y}_{\bar{\imath}}^\mu \bar{X}_\nu^{\bar{\imath}} = \delta^\mu{}_\nu$ hold respectively, and the B-field becomes irrelevant. In these extremely non-Riemannian limits, the DFT metric coincides with the $\mathbf{O}(D,D)$-invariant metric up to sign,

$$
\mathcal{H}_{AB} = \pm\mathcal{J}_{AB} = \left(\begin{array}{cc} 0 & \pm 1 \\ \pm 1 & 0 \end{array}\right)\,. \tag{2.34}
$$

These cases represent two perfectly $\mathbf{O}(D,D)$-symmetric vacua in bosonic DFT that are maximally non-Riemannian, characterized by the saturation, $n + \bar{n} = D$. Remarkably, these vacua exhibit infinitely many isometries [47], as any local parameter ξ^A automatically satisfies the Killing equation, defined by the condition $\hat{\mathcal{L}}_\xi\mathcal{H}_{AB} = 0$.

In this scenario, the ordinary Riemannian spacetime (2.33) emerges after spontaneous symmetry breaking of the fully $\mathbf{O}(D,D)$-symmetric vacua (2.34), while the Riemannian metric and the B-field are identified as Nambu–Goldstone bosons [48, 49].

On a generic $(n,\bar{n})$ non-Riemannian background, a particle described by the doubled action (2.14) freezes in $n+\bar{n}$ untilde directions,

$$X^{i}_{\mu}\frac{\mathrm{d}x^{\mu}}{\mathrm{d}\tau}=0\,,\qquad\qquad \bar{X}^{\bar{\imath}}_{\mu}\frac{\mathrm{d}x^{\mu}}{\mathrm{d}\tau}=0\,,\tag{2.35}$$

while the string in (2.15) becomes chiral and anti-chiral in the n and $\bar{n}$ untilde directions, respectively:

$$X^{i}_{\mu}\left(\partial_{\alpha}x^{\mu}+\tfrac{1}{\sqrt{-h}}\epsilon_{\alpha}{}^{\beta}\partial_{\beta}x^{\mu}\right)=0\,,\qquad\qquad \bar{X}^{\bar{\imath}}_{\mu}\left(\partial_{\alpha}x^{\mu}-\tfrac{1}{\sqrt{-h}}\epsilon_{\alpha}{}^{\beta}\partial_{\beta}x^{\mu}\right)=0\,.\tag{2.36}$$

This behaviour arises as the components of the auxiliary gauge potential $\mathfrak{a}^{A}$ act as Lagrange multipliers [44].

Nonetheless, the $(n,\bar{n})$-classification of the generalised metric (2.26) described above remains valid for bosonic DFT geometries. In the presence of fermions, particularly in supersymmetric DFTs [33, 34], the twofold spin groups are *a priori* fixed, and the permissible values of $(n,\bar{n})$ are constrained by their dimensions and signatures. Specifically, the Euclidean spin group, $\mathbf{Spin}(D)\times\mathbf{Spin}(D)$, does not permit any non-Riemannian geometries, whereas the Minkowskian case, $\mathbf{Spin}(1,D-1)\times\mathbf{Spin}(D-1,1)$, as assumed in Table 1, allows $(1,1)$ non-Riemannian geometry [40, 50], which corresponds to non-relativistic [51–53] or Newton–Cartan string theories [54–63]. Furthermore, certain known singularities in supergravity theories can be appropriately identified as regular $(1,1)$ non-Riemannian geometries [64]; see section 3.1 for an example.

While we refer to [44] for the general $(n,\bar{n})$ cases, here we merely spell out the $(0,0)$ Riemannian parametrisation of the DFT vielbeins,

$$V_{Ap}=\tfrac{1}{\sqrt{2}}\left(\begin{array}{c}e_{p}{}^{\mu}\\[2ex] e_{\nu}{}^{q}\eta_{qp}+B_{\nu\sigma}e_{p}{}^{\sigma}\end{array}\right),\qquad \bar{V}_{A\bar{p}}=\tfrac{1}{\sqrt{2}}\left(\begin{array}{c}\bar{e}_{\bar{p}}{}^{\mu}\\[2ex] \bar{e}_{\nu}{}^{\bar{q}}\bar{\eta}_{\bar{q}\bar{p}}+B_{\nu\sigma}\bar{e}_{\bar{p}}{}^{\sigma}\end{array}\right).\tag{2.37}$$

Here $e_{\mu}{}^{p}$ and $\bar{e}_{\mu}{}^{\bar{p}}$ are a pair of (Riemannian) vielbeins which square to the same metric,

$$e_{\mu}{}^{p}e_{\nu p}=-\bar{e}_{\mu}{}^{\bar{p}}\bar{e}_{\nu\bar{p}}=g_{\mu\nu}\,.\tag{2.38}$$

Their dual presence allows $\mathbf{O}(D,D)$ to safely rotate the capital-letter indices of the vielbeins exclusively [65].

Lastly, the DFT dilaton is parametrised by the Riemannian metric and the (weightless) string dilaton:

$$e^{-2d}=\sqrt{-g}\,e^{-2\phi}\,.\tag{2.39}$$

Summary. DFT geometries are classified into Riemannian and non-Riemannian cases, unifying stringy spacetime structures. Ordinary spacetime appears as the $(0,0)$ case, while maximally $\mathbf{O}(D,D)$-symmetric vacua are non-Riemannian and chiral.

2.4 Christoffel Symbols & Spin Connections

Aim. To build a master derivative that unifies diffeomorphisms and local Lorentz symmetries.

The master derivative, denoted as $\mathcal{D}_A$, is introduced to unify and covariantly describe all the local symmetries: the doubled-yet-gauged diffeomorphisms and the twofold local Lorentz symmetries, $\mathbf{Spin}(1,D{-}1)\times\mathbf{Spin}(D{-}1,1)$. More explicitly, the master derivative is postulated as:

$$\mathcal{D}_A = \partial_A + \Gamma_A + \Phi_A + \bar{\Phi}_A\,. \tag{2.40}$$

Here, Γ_A refers to the Christoffel connection for the doubled-yet-gauged diffeomorphisms, introduced in [65] and based on earlier work [29].[4,5] The connection is defined as:

$$\begin{aligned}\Gamma_{CAB} = \;& 2\left(P\partial_C P\bar{P}\right)_{[AB]} + 2\left(\bar{P}_{[A}{}^{D}\bar{P}_{B]}{}^{E} - P_{[A}{}^{D}P_{B]}{}^{E}\right)\partial_D P_{EC} \\ & -4\left(\tfrac{1}{P_M{}^M-1}P_{C[A}P_{B]}{}^{D} + \tfrac{1}{\bar{P}_M{}^M-1}\bar{P}_{C[A}\bar{P}_{B]}{}^{D}\right)\left(\partial_D d + (P\partial^E P\bar{P})_{[ED]}\right)\,.\end{aligned} \tag{2.41}$$

Additionally, Φ_A and $\bar{\Phi}_A$ are the spin connections corresponding to the twofold local Lorentz symmetries [67], given as:

$$\Phi_{Apq} = \Phi_{A[pq]} = V^B{}_p\nabla_A V_{Bq}\,, \qquad \bar{\Phi}_{A\bar{p}\bar{q}} = \bar{\Phi}_{A[\bar{p}\bar{q}]} = \bar{V}^B{}_{\bar{p}}\nabla_A\bar{V}_{B\bar{q}}\,. \tag{2.42}$$

In our notation, ∇_A is the diffeomorphism-covariant derivative that involves the Christoffel symbols only,

$$\nabla_A := \partial_A + \Gamma_A\,, \tag{2.43}$$

and, ignoring any local Lorentz indices, acts on a tensor density with weight ω as:

$$\nabla_C T_{A_1A_2\cdots A_n} := \partial_C T_{A_1A_2\cdots A_n} - \omega_T\,\Gamma^B{}_{BC}T_{A_1A_2\cdots A_n} + \sum_{i=1}^{n}\Gamma_{CA_i}{}^{B}T_{A_1\cdots A_{i-1}BA_{i+1}\cdots A_n}\,. \tag{2.44}$$

In particular, in (2.42),

$$\nabla_A V_{Bq} = \partial_A V_{Bq} + \Gamma_{AB}{}^{C}V_{Cq}\,, \qquad \nabla_A\bar{V}_{B\bar{q}} = \partial_A\bar{V}_{B\bar{q}} + \Gamma_{AB}{}^{C}\bar{V}_{C\bar{q}}\,. \tag{2.45}$$

The Christoffel connection (2.41) is uniquely fixed by requiring the following three properties:

---

[4] One key lesson from [29] is that the generalised metric alone is insufficient for fully constructing covariant derivatives and curvatures; the inclusion of the dilaton d is essential.

[5] An alternative formulation was proposed in [3] and developed in [66], where curved $\mathbf{O}(D,D)$ indices are mapped to flat Lorentz indices via the vielbeins (2.21). Although this appears to bypass the Christoffel connection, the spin connection necessarily reintroduces its content, as in (2.42), so that the framework ultimately preserves parallels with conventional differential geometry.

i) Full compatibility with all fundamental fields,

$$\mathcal{D}_A P_{BC} = \nabla_A P_{BC} = 0\,, \qquad \mathcal{D}_A \bar{P}_{BC} = \nabla_A \bar{P}_{BC} = 0\,,$$
$$\mathcal{D}_A d\; = \nabla_A d\; = -\tfrac{1}{2}e^{2d}\nabla_A(e^{-2d}) = \partial_A d + \tfrac{1}{2}\Gamma^B{}_{BA} = 0\,, \tag{2.46}$$

which implies

$$\mathcal{D}_A \mathcal{J}_{BC} = \nabla_A \mathcal{J}_{BC} = 0\,, \qquad \mathcal{D}_A \mathcal{H}_{BC} = \nabla_A \mathcal{H}_{BC} = 0\,, \tag{2.47}$$

such that Γ_A is, as expected, $\mathbf{so}(D,D)$-valued:

$$\Gamma_{ABC} = -\Gamma_{ACB}\,. \tag{2.48}$$

ii) Torsionless cyclic property,[6]

$$\Gamma_{ABC} + \Gamma_{BCA} + \Gamma_{CAB} = 0\,, \tag{2.49}$$

which makes ∇_A compatible with the generalised Lie derivative (2.5) and the C-bracket (2.7) so that ordinary derivatives in these expressions can be freely replaced by ∇_A:

$$\hat{\mathcal{L}}_\xi T_{A_1\cdots A_n} = \xi^B\nabla_B T_{A_1\cdots A_n} + \omega\nabla_B\xi^B T_{A_1\cdots A_n} + \sum_{i=1}^{n}(\nabla_{A_i}\xi_B - \nabla_B\xi_{A_i})T_{A_1\cdots A_{i-1}}{}^B{}_{A_{i+1}\cdots A_n}\,. \tag{2.50}$$

iii) Projection constraints,

$$\mathcal{P}_{ABC}{}^{DEF}\Gamma_{DEF} = 0\,, \qquad \bar{\mathcal{P}}_{ABC}{}^{DEF}\Gamma_{DEF} = 0\,. \tag{2.51}$$

These constraints involve six-index projectors,

$$\mathcal{P}_{ABC}{}^{DEF} := P_A{}^D P_{[B}{}^{[E}P_{C]}{}^{F]} + \tfrac{2}{P_M{}^M-1}P_{A[B}P_{C]}{}^{[E}P^{F]D}\,,$$
$$\bar{\mathcal{P}}_{ABC}{}^{DEF} := \bar{P}_A{}^D \bar{P}_{[B}{}^{[E}\bar{P}_{C]}{}^{F]} + \tfrac{2}{\bar{P}_M{}^M-1}\bar{P}_{A[B}\bar{P}_{C]}{}^{[E}\bar{P}^{F]D}\,, \tag{2.52}$$

which satisfy the following projection properties,

$$\mathcal{P}_{ABC}{}^{DEF}\mathcal{P}_{DEF}{}^{GHI} = \mathcal{P}_{ABC}{}^{GHI}\,, \qquad \bar{\mathcal{P}}_{ABC}{}^{DEF}\bar{\mathcal{P}}_{DEF}{}^{GHI} = \bar{\mathcal{P}}_{ABC}{}^{GHI}\,, \tag{2.53}$$

[6] In full-order supersymmetric DFTs [33, 34], the torsions are quadratic in fermionic fields and naturally implement the 1.5 formalism characteristic of supergravity theories.

various symmetric relations [68],

$$\begin{array}{lll}\mathcal{P}_{ABCDEF}=\mathcal{P}_{DEFABC}\,, & \mathcal{P}_{ABCDEF}=\mathcal{P}_{A[BC]D[EF]}\,, & \mathcal{P}_{[AB]CDEF}=\mathcal{P}_{CAB[EF]D}\,,\\ \bar{\mathcal{P}}_{ABCDEF}=\bar{\mathcal{P}}_{DEFABC}\,, & \bar{\mathcal{P}}_{ABCDEF}=\bar{\mathcal{P}}_{A[BC]D[EF]}\,, & \bar{\mathcal{P}}_{[AB]CDEF}=\bar{\mathcal{P}}_{CAB[EF]D}\,,\end{array} \tag{2.54}$$

and traceless conditions,

$$P^{AB}\mathcal{P}_{ABCDEF}=0\,,\qquad\qquad \bar{P}^{AB}\bar{\mathcal{P}}_{ABCDEF}=0\,. \tag{2.55}$$

Unlike the GR Christoffel symbols, there exist no normal coordinates where the DFT Christoffel symbols would vanish pointwise. The Equivalence Principle holds for point particles but not for extended objects such as strings [35]. Parallel to the variation of the Christoffel symbols in GR:

$$\delta\gamma^{\lambda}_{\mu\nu}=\frac{1}{2}(\bigtriangledown_{\mu}\delta g^{\lambda}{}_{\nu}+\bigtriangledown_{\nu}\delta g_{\mu}{}^{\lambda}-\bigtriangledown^{\lambda}\delta g_{\mu\nu})\,,$$

the DFT Christoffel symbols vary infinitesimally by the DFT metric and dilaton: with $\delta P_{AB}=\frac{1}{2}\delta\mathcal{H}_{AB}$,

$$\begin{array}{rll}\delta\Gamma_{CAB} & = & 2P_{[A}^{~D}\bar{P}_{B]}^{~E}\nabla_{C}\delta P_{DE}+2(\bar{P}_{[A}^{~D}\bar{P}_{B]}^{~E}-P_{[A}^{~D}P_{B]}^{~E})\nabla_{D}\delta P_{EC}\\ & & -\frac{4}{D-1}(\bar{P}_{C[A}\bar{P}_{B]}^{~D}+P_{C[A}P_{B]}^{~D})(\partial_{D}\delta d+P_{E[G}\nabla^{G}\delta P^{E}_{~D]})-\Gamma_{FDE}\,\delta(\mathcal{P}+\bar{\mathcal{P}})_{CAB}{}^{FDE}\,.\end{array} \tag{2.56}$$

Once the DFT Christoffel symbols are fixed, the spin connections (2.42) follow naturally from the compatibility of the master derivative with the DFT vielbeins: from (2.45),

$$\mathcal{D}_{A}V_{Bp}=\nabla_{A}V_{Bp}+\Phi_{Ap}{}^{q}V_{Bq}=0\,,\qquad\qquad \mathcal{D}_{A}\bar{V}_{B\bar{p}}=\nabla_{A}\bar{V}_{B\bar{p}}+\bar{\Phi}_{A\bar{p}}{}^{\bar{q}}\bar{V}_{B\bar{q}}=0\,. \tag{2.57}$$

The master derivative is also compatible with the metrics and gamma matrices of the twofold spin groups,

$$\mathcal{D}_{A}\eta_{pq}=0\,,\qquad \mathcal{D}_{A}\bar{\eta}_{\bar{p}\bar{q}}=0\,,\qquad \mathcal{D}_{A}(\gamma^{p})^{\alpha}{}_{\beta}=0\,,\qquad \mathcal{D}_{A}(\bar{\gamma}^{\bar{p}})^{\bar{\alpha}}{}_{\bar{\beta}}=0\,, \tag{2.58}$$

and thus, analogous to GR,

$$\Phi_{Apq}=-\Phi_{Aqp}\,,\qquad \bar{\Phi}_{A\bar{p}\bar{q}}=-\bar{\Phi}_{A\bar{q}\bar{p}}\,,\qquad \Phi_{A}{}^{\alpha}{}_{\beta}=\tfrac{1}{4}\Phi_{Apq}(\gamma^{pq})^{\alpha}{}_{\beta}\,,\qquad \bar{\Phi}_{A}{}^{\bar{\alpha}}{}_{\bar{\beta}}=\tfrac{1}{4}\bar{\Phi}_{A\bar{p}\bar{q}}(\bar{\gamma}^{\bar{p}\bar{q}})^{\bar{\alpha}}{}_{\bar{\beta}}\,. \tag{2.59}$$

It is worthwhile to note a formula that relates the three connections [68]:

$$\Gamma_{CAB} = V_A{}^p\partial_C V_{Bp} + \bar{V}_A{}^{\bar{p}}\partial_C\bar{V}_{B\bar{p}} + V_A{}^pV_B{}^q\Phi_{Cpq} + \bar{V}_A{}^{\bar{p}}\bar{V}_B{}^{\bar{q}}\bar{\Phi}_{C\bar{p}\bar{q}}\,, \tag{2.60}$$

ensuring that Cartan's structure equations in GR hold analogously in DFT [68].

Due to the section condition, the Christoffel symbols satisfy

$$P_A{}^C\bar{P}_B{}^D\Gamma^E{}_{CD}\partial_E = 0\,. \tag{2.61}$$

Summary. The Christoffel and spin connections in DFT are uniquely determined by compatibility and extra (torsionless) conditions, extending GR into the doubled framework.

2.5 From Semi-Covariance to Full Covariance

Aim. To refine the semi-covariant derivative into a fully covariant derivative through projections.

Explicitly, the master derivative (2.40) acts as

$$\begin{aligned}&\mathcal{D}_AT_{Bp}{}^{\alpha}{}_{\bar{p}}{}^{\bar{\alpha}} = \nabla_AT_{Bp}{}^{\alpha}{}_{\bar{p}}{}^{\bar{\alpha}} + \Phi_{Ap}{}^qT_{Bq}{}^{\alpha}{}_{\bar{p}}{}^{\bar{\alpha}} + \Phi_A{}^{\alpha}{}_{\beta}T_{Bp}{}^{\beta}{}_{\bar{p}}{}^{\bar{\alpha}} + \bar{\Phi}_{A\bar{p}}{}^{\bar{q}}T_{Bp}{}^{\alpha}{}_{\bar{q}}{}^{\bar{\alpha}} + \bar{\Phi}_A{}^{\bar{\alpha}}{}_{\bar{\beta}}T_{Bp}{}^{\alpha}{}_{\bar{p}}{}^{\bar{\beta}}\\ &= \partial_AT_{Bp}{}^{\alpha}{}_{\bar{p}}{}^{\bar{\alpha}} - \omega\Gamma^C{}_{CA}T_{Bp}{}^{\alpha}{}_{\bar{p}}{}^{\bar{\alpha}} + \Gamma_{AB}{}^CT_{Cp}{}^{\alpha}{}_{\bar{p}}{}^{\bar{\alpha}} + \Phi_{Ap}{}^qT_{Bq}{}^{\alpha}{}_{\bar{p}}{}^{\bar{\alpha}} + \Phi_A{}^{\alpha}{}_{\beta}T_{Bp}{}^{\beta}{}_{\bar{p}}{}^{\bar{\alpha}} + \bar{\Phi}_{A\bar{p}}{}^{\bar{q}}T_{Bp}{}^{\alpha}{}_{\bar{q}}{}^{\bar{\alpha}} + \bar{\Phi}_A{}^{\bar{\alpha}}{}_{\bar{\beta}}T_{Bp}{}^{\alpha}{}_{\bar{p}}{}^{\bar{\beta}}\,.\end{aligned} \tag{2.62}$$

The master derivative is fully covariant with respect to the twofold local Lorentz symmetries but is, *a priori*, only *semi-covariant* under the doubled-yet-gauged diffeomorphisms. To achieve complete covariance under these diffeomorphisms, an additional step of projection is required, which we explain.

✓ Tensorial Covariant Derivatives

Aim. To obtain a fully covariant derivative for tensors.

Under diffeomorphisms, the DFT Christoffel symbols transform as

$$\delta_\xi\Gamma_{CAB} = \hat{\mathcal{L}}_\xi\Gamma_{CAB} - 2\partial_C\partial_{[A}\xi_{B]} + 2(\mathcal{P}+\bar{\mathcal{P}})_{CAB}{}^{DEF}\partial_D\partial_{[E}\xi_{F]}\,, \tag{2.63}$$

such that ∇_A, and consequently $\mathcal{D}_A$, are not inherently covariant under the diffeomorphisms:

$$\delta_\xi\big(\nabla_CT_{A_1\cdots A_n}\big) = \hat{\mathcal{L}}_\xi\big(\nabla_CT_{A_1\cdots A_n}\big) + \sum_{i=1}^{n}2(\mathcal{P}+\bar{\mathcal{P}})_{CA_i}{}^{BDEF}\partial_D\partial_E\xi_F\,T_{A_1\cdots A_{i-1}BA_{i+1}\cdots A_n}\,. \tag{2.64}$$

However, the anomalous terms consistently appear through the six-index projectors (2.52), making them straightforward to project out. For instance, in (2.64), projecting the derivative index and the tensor indices in opposite ways allows the construction of fully covariant derivatives [65]:

$$P_C{}^D \bar{P}_{A_1}{}^{B_1} \cdots \bar{P}_{A_n}{}^{B_n} \nabla_D T_{B_1 \cdots B_n} \,, \qquad \bar{P}_C{}^D P_{A_1}{}^{B_1} \cdots P_{A_n}{}^{B_n} \nabla_D T_{B_1 \cdots B_n} \,. \tag{2.65}$$

Similarly, using (2.54) and (2.55), we obtain fully covariant *divergences*,

$$P^{AB} \bar{P}_{C_1}{}^{D_1} \cdots \bar{P}_{C_n}{}^{D_n} \nabla_A T_{BD_1 \cdots D_n} \,, \qquad \bar{P}^{AB} P_{C_1}{}^{D_1} \cdots P_{C_n}{}^{D_n} \nabla_A T_{BD_1 \cdots D_n} \,. \tag{2.66}$$

For instance, the divergence of a weightless vector, J^A, reads

$$\nabla_A \Big(e^{-2d} J^A \Big) = \partial_A \Big(e^{-2d} J^A \Big) = e^{-2d} \nabla_A J^A = e^{-2d} \left(P^{AB} + \bar{P}^{AB} \right) \nabla_A J_B \,. \tag{2.67}$$

Nevertheless, when acting on the two-index projectors, P_{AB} and $\bar{P}_{AB}$, the anomalous terms in (2.64) are automatically eliminated due to the properties of the six-index projectors, (2.54) and (2.55). This guarantees the full covariance of the compatibility condition, $\nabla_C P_{AB} = 0 = \nabla_C \bar{P}_{AB}$, as postulated in (2.46).

In accordance with (2.24), the partial derivative of the generalised metric satisfies a projection property,

$$\partial_C \mathcal{H}_{AB} = (P \partial_C \mathcal{H} \bar{P})_{AB} + (\bar{P} \partial_C \mathcal{H} P)_{AB} \,, \tag{2.68}$$

where the two free indices are projected in opposite manners. This property essentially precludes the possibility of canceling the anomalous terms in (2.63) by adding any derivatives of the generalised metric or dilaton to Γ_{CAB} in (2.41). While introducing additional complementary fields could, in principle, eliminate the anomalous terms, such fields remain "undetermined" or are not naturally identifiable within string theory.

Notably, the additional projection step (2.65) highlights the more refined and rigid structure of DFT compared to GR. For instance, while DFT provides two available "metrics," namely $\mathcal{J}_{AB}$ and $\mathcal{H}_{AB}$, the pairwise contraction of $\mathbf{O}(D,D)$ indices is significantly restricted. Specifically, when constructing a scalar from the squared derivative $\nabla_A T_{B_1 B_2 \cdots B_n}$ for a kinetic term, from (2.65) there are only two viable ways to "square" it, rather than the 2^{n+1} possibilities that might otherwise exist.

As a further illustration, consider a doubled Yang–Mills potential $\mathbf{A}_A$. The fully covariant skew-symmetric field strength is expressed as [35, 69]

$$P_A{}^C \bar{P}_B{}^D \mathbf{F}_{CD} = P_A{}^C \bar{P}_B{}^D \big(\nabla_C \mathbf{A}_D - \nabla_D \mathbf{A}_C - i \left[\mathbf{A}_C, \mathbf{A}_D \right] \big) \,, \tag{2.69}$$

and there is only one way to construct the doubled Yang–Mills action:

$$\int_{\Sigma_D} e^{-2d} P^{AC} \bar{P}^{BD} \mathrm{Tr}(\mathbf{F}_{AB}\mathbf{F}_{CD}) \,, \tag{2.70}$$

where Σ_D is a D-dimensional section over which the integral is taken with the measure, e^{-2d}.

The doubled Yang–Mills potential decomposes into a displacement vector and an ordinary one-form: $\mathbf{A}_A = (\varphi^\mu, A_\nu)$. This structure enables the doubled Yang–Mills action to provide a unified description of gluons/photons and (non-Abelian) phonons [70]. However, it is always possible to eliminate the displacement phonon vector by imposing an $\mathbf{O}(D, D)$-symmetric constraint, $\mathbf{A}^A \partial_A = 0$, analogous to the section condition (2.1) [35].

We proceed to construct fully covariant second-order differential operators, for which we need to identify the relevant anomalous terms. Making use of (2.64) repeatedly with care, we get

$$\begin{array}{rcl} \big(\delta_\xi - \hat{\mathcal{L}}_\xi\big)\big(\nabla_B \nabla_C T_{A_1\cdots A_n}\big) & = & 2(\mathcal{P}{+}\bar{\mathcal{P}})_{BC}{}^{DEFG}\partial_E\partial_F\xi_G \, \nabla_D T_{A_1\cdots A_n} \\ & & +\displaystyle\sum_{i=1}^{n} \left[\begin{array}{l} 2(\mathcal{P}{+}\bar{\mathcal{P}})_{CA_i}{}^{DEFG} T_{A_1\cdots A_{i-1}DA_{i+1}\cdots A_n} \nabla_B \left(\partial_E\partial_F\xi_G\right) \\ +2(\mathcal{P}{+}\bar{\mathcal{P}})_{CA_i}{}^{DEFG}\partial_E\partial_F\xi_G \, \nabla_B T_{A_1\cdots A_{i-1}DA_{i+1}\cdots A_n} \\ +2(\mathcal{P}{+}\bar{\mathcal{P}})_{BA_i}{}^{DEFG}\partial_E\partial_F\xi_G \, \nabla_C T_{A_1\cdots A_{i-1}DA_{i+1}\cdots A_n} \end{array}\right] . \end{array} \tag{2.71}$$

Again from the symmetric and traceless relations of the six-index projectors (2.54), (2.55), it is straightforward to obtain fully covariant *d'Alembertians*:

$$P^{AB}\bar{P}_{C_1}{}^{D_1}\cdots\bar{P}_{C_n}{}^{D_n}\nabla_A\nabla_B T_{D_1\cdots D_n} \,, \qquad\qquad \bar{P}^{AB}P_{C_1}{}^{D_1}\cdots P_{C_n}{}^{D_n}\nabla_A\nabla_B T_{D_1\cdots D_n} \,. \tag{2.72}$$

Later, through equations (2.128), (2.129), (2.130), and (2.131), we will encounter a pair of general forms of the d'Alembertian, which yield a box operator. These operators are crafted to act on arbitrary multi-index tensor densities while *a priori* incorporating four-index (Riemann) curvature.

Summary. Projections remove anomalous terms and enforce strict covariance, showing that DFT is structurally more rigid than GR.

✓ Spinorial Covariant Derivatives

Aim. To extend covariance to spinors and the RR sector by constructing *Dirac operators*.

It is possible to freely replace ∇_A in the fully covariant derivatives (2.65), (2.66), and (2.72) with the master derivative $\mathcal{D}_A$, while simultaneously contracting their projected, otherwise unconstrained $\mathbf{O}(D,D)$ vector indices using the DFT vielbeins, V_{Ap} and $\bar{V}_{A\bar{q}}$. Due to the compatibility of the DFT vielbeins with the master derivative (2.57), this replacement reduces the fully covariant derivatives to more streamlined forms:

$$\mathcal{D}_p T_{\bar{q}_1\cdots\bar{q}_n}\,,\qquad \mathcal{D}_{\bar{p}} T_{q_1\cdots q_n}\,,\qquad \mathcal{D}_p T^p{}_{\bar{q}_1\cdots\bar{q}_n}\,,\qquad \mathcal{D}_{\bar{p}} T^{\bar{p}}{}_{q_1\cdots q_n}\,,\qquad \mathcal{D}_p\mathcal{D}^p T_{\bar{q}_1\cdots\bar{q}_n}\,,\qquad \mathcal{D}_{\bar{p}}\mathcal{D}^{\bar{p}} T_{q_1\cdots q_n}\,, \tag{2.73}$$

where $\mathcal{D}_p = V^A{}_p\mathcal{D}_A$ and $\mathcal{D}_{\bar{p}} = \bar{V}^A{}_{\bar{p}}\mathcal{D}_A$.

The fully covariant doubled Yang–Mills field strength (2.69) also reads

$$\mathbf{F}_{p\bar{q}} := V^A{}_p\bar{V}^B{}_{\bar{q}}\big(\nabla_A\mathbf{A}_B - \nabla_B\mathbf{A}_A - i\,[\mathbf{A}_A,\mathbf{A}_B]\big) = \mathcal{D}_p\mathbf{A}_{\bar{q}} - \mathcal{D}_{\bar{q}}\mathbf{A}_p - i\,[\mathbf{A}_p,\mathbf{A}_{\bar{q}}]\;. \tag{2.74}$$

The full covariance of (2.73) and (2.74) can also be directly confirmed by observing that, from

$$\delta_\xi\Phi_{Apq} = \hat{\mathcal{L}}_\xi\Phi_{Apq} + 2\mathcal{P}_{Apq}{}^{DEF}\partial_D\partial_{[E}\xi_{F]}\,,\qquad\qquad \delta_\xi\bar{\Phi}_{A\bar{p}\bar{q}} = \hat{\mathcal{L}}_\xi\bar{\Phi}_{A\bar{p}\bar{q}} + 2\bar{\mathcal{P}}_{A\bar{p}\bar{q}}{}^{DEF}\partial_D\partial_{[E}\xi_{F]}\,, \tag{2.75}$$

the following projected components of the spin connections are fully covariant under doubled-yet-gauged diffeomorphisms:[7]

$$\Phi_{\bar{r}pq}\,,\qquad \bar{\Phi}_{r\bar{p}\bar{q}}\,,\qquad \Phi_{[pqr]}\,,\qquad \bar{\Phi}_{[\bar{p}\bar{q}\bar{r}]}\,,\qquad \Phi^p{}_{pq}\,,\qquad \bar{\Phi}^{\bar{p}}{}_{\bar{p}\bar{q}}\,, \tag{2.76}$$

where we have set $\Phi_{\bar{r}pq} = \bar{V}^A{}_{\bar{r}}\Phi_{Apq}$ and $\Phi_{rpq} = V^A{}_r\Phi_{Apq}$, *etc.*

Consequently, when acting on $\mathbf{Spin}(1,D{-}1)$ spinors, such as ρ^α or $\psi^\alpha_{\bar{p}}$, or on $\mathbf{Spin}(D{-}1,1)$ spinors, such as $\rho'^{\bar{\alpha}}$ or $\psi'^{\bar{\alpha}}_p$, the fully covariant *Dirac operators* are [33, 67]

$$\gamma^p\mathcal{D}_p\rho\,,\qquad \gamma^p\mathcal{D}_p\psi_{\bar{p}}\,,\qquad \mathcal{D}_{\bar{p}}\rho\,,\qquad \mathcal{D}_{\bar{p}}\psi^{\bar{p}}\,,\qquad \bar{\gamma}^{\bar{p}}\mathcal{D}_{\bar{p}}\rho'\,,\qquad \bar{\gamma}^{\bar{p}}\mathcal{D}_{\bar{p}}\psi'_p\,,\qquad \mathcal{D}_p\rho'\,,\qquad \mathcal{D}_p\psi'^p\,. \tag{2.77}$$

Further, in the maximally supersymmetric type II DFT [34], as well as in the pure spinor formalism [71], the Ramond–Ramond (RR) sector is characterized by a $\mathbf{Spin}(1,D{-}1)\times\mathbf{Spin}(D{-}1,1)$ bi-fundamental spinorial potential: $\mathcal{C}^\alpha{}_{\bar{\alpha}}$ (*c.f.* [72–77]). Subsequently, a pair of fully covariant and nilpotent derivatives, $\mathcal{D}_+$ and $\mathcal{D}_-$, are introduced [78]:

$$\mathcal{D}_\pm\mathcal{C} := \gamma^p\mathcal{D}_p\mathcal{C} \pm \gamma^{(D+1)}\mathcal{D}_{\bar{p}}\mathcal{C}\bar{\gamma}^{\bar{p}}\,,\qquad\qquad \mathcal{D}^2_\pm\mathcal{C} = 0\,, \tag{2.78}$$

[7] The quantities in (2.76) are essentially the "generalised fluxes" considered in [79–81] as the building blocks of DFT.

where, with (2.59), $\mathcal{D}_A\mathcal{C} = \partial_A\mathcal{C} + \Phi_A\mathcal{C} - \mathcal{C}\bar{\Phi}_A$. In particular, the RR field strength, $\mathcal{F}^{\alpha}{}_{\bar{\alpha}}$, is given by one of these operators and remains invariant under RR gauge transformations,

$$\mathcal{F} := \mathcal{D}_+\mathcal{C}\,, \qquad\qquad \delta\mathcal{C} = \mathcal{D}_+\lambda \quad\longrightarrow\quad \delta\mathcal{F} = 0\,. \tag{2.79}$$

✓ Generalised Kosmann Derivative, aka the Further-Generalised Lie Derivative

Aim. To adapt the Lie derivative so that it respects local Lorentz symmetry.

The generalised Lie derivative is compatible with the semi-covariant derivative, $\hat{\mathcal{L}}_\xi(\partial_A) = \hat{\mathcal{L}}_\xi(\nabla_A)$, as demonstrated in (2.50), and is in fact fully covariant under the doubled-yet-gauged diffeomorphisms (2.8). However, when acting on spinorial tensors, it fails to preserve the local Lorentz symmetries. To achieve full covariance under both diffeomorphisms and local Lorentz rotations, the generalised Lie derivative must be further extended to incorporate spin connections. This approach was originally introduced in the context of GR by Kosmann in 1971 [82].[8] The DFT counterpart to the Kosmann derivative, also referred to as the further-generalised Lie derivative, is defined in [25] as

$$\begin{aligned}
\widetilde{\mathcal{L}}_\xi T_{Ap\bar{p}}{}^{\alpha}{}_{\beta}{}^{\bar{\alpha}}{}_{\bar{\beta}} \;\;:=\;\; & \xi^B\mathcal{D}_B T_{Ap\bar{p}}{}^{\alpha}{}_{\beta}{}^{\bar{\alpha}}{}_{\bar{\beta}} + \omega\mathcal{D}_B\xi^B T_{Ap\bar{p}}{}^{\alpha}{}_{\beta}{}^{\bar{\alpha}}{}_{\bar{\beta}} + 2\mathcal{D}_{[A}\xi_{B]}T^B{}_{p\bar{p}}{}^{\alpha}{}_{\beta}{}^{\bar{\alpha}}{}_{\bar{\beta}} \\
& + 2\mathcal{D}_{[p}\xi_{q]}T_A{}^q{}_{\bar{p}}{}^{\alpha}{}_{\beta}{}^{\bar{\alpha}}{}_{\bar{\beta}} + \tfrac{1}{2}\mathcal{D}_{[r}\xi_{s]}(\gamma^{rs})^{\alpha}{}_{\delta}T_{Ap\bar{p}}{}^{\delta}{}_{\beta}{}^{\bar{\alpha}}{}_{\bar{\beta}} - \tfrac{1}{2}\mathcal{D}_{[r}\xi_{s]}(\gamma^{rs})^{\delta}{}_{\beta}T_{Ap\bar{p}}{}^{\alpha}{}_{\delta}{}^{\bar{\alpha}}{}_{\bar{\beta}} \\
& + 2\mathcal{D}_{[\bar{p}}\xi_{\bar{q}]}T_{Ap}{}^{\bar{q}\alpha}{}_{\beta}{}^{\bar{\alpha}}{}_{\bar{\beta}} + \tfrac{1}{2}\mathcal{D}_{[\bar{r}}\xi_{\bar{s}]}(\bar{\gamma}^{\bar{r}\bar{s}})^{\bar{\alpha}}{}_{\bar{\delta}}T_{Ap\bar{p}}{}^{\alpha}{}_{\beta}{}^{\bar{\delta}}{}_{\bar{\beta}} - \tfrac{1}{2}\mathcal{D}_{[\bar{r}}\xi_{\bar{s}]}(\bar{\gamma}^{\bar{r}\bar{s}})^{\bar{\delta}}{}_{\bar{\beta}}T_{Ap\bar{p}}{}^{\alpha}{}_{\beta}{}^{\bar{\alpha}}{}_{\bar{\delta}}\,,
\end{aligned} \tag{2.80}$$

which consists of the generalised Lie derivative combined with infinitesimal local Lorentz rotations characterized by

$$\xi^A\Phi_{Apq} + 2\mathcal{D}_{[p}\xi_{q]} = 2\partial_{[p}\xi_{q]} + \Phi_{\bar{r}pq}\xi^{\bar{r}} + 3\Phi_{[pqr]}\xi^r\,, \qquad \xi^A\bar{\Phi}_{A\bar{p}\bar{q}} + 2\mathcal{D}_{[\bar{p}}\xi_{\bar{q}]} = 2\partial_{[\bar{p}}\xi_{\bar{q}]} + \bar{\Phi}_{r\bar{p}\bar{q}}\xi^r + 3\bar{\Phi}_{[\bar{p}\bar{q}\bar{r}]}\xi^{\bar{r}}\,. \tag{2.81}$$

As listed in (2.76), all of these are fully covariant under the doubled-yet-gauged diffeomorphisms.

✓ Riemannian Parametrisation of the Covariant Derivatives

Aim. To show how covariant derivatives reduce under the $(0,0)$ Riemannian parametrisation.

[8] See [83] for a recent application of the Kosmann derivative.

Under the $(0,0)$ Riemannian parametrisation, (2.37) and (2.39), the projected components of the spin connections (2.76) reduce explicitly to

$$\begin{aligned}
\Phi_{\bar{p}pq} &= \tfrac{1}{\sqrt{2}}\bar{e}_{\bar{p}}{}^{\mu}\left(\omega_{\mu pq} + \tfrac{1}{2}H_{\mu pq}\right) , &\qquad \bar{\Phi}_{p\bar{p}\bar{q}} &= \tfrac{1}{\sqrt{2}}e_{p}{}^{\mu}\left(\bar{\omega}_{\mu\bar{p}\bar{q}} + \tfrac{1}{2}H_{\mu\bar{p}\bar{q}}\right) , \\
\Phi_{[pqr]} &= \tfrac{1}{\sqrt{2}}\left(\omega_{[pqr]} + \tfrac{1}{6}H_{pqr}\right) , &\qquad \bar{\Phi}_{[\bar{p}\bar{q}\bar{r}]} &= \tfrac{1}{\sqrt{2}}\left(\bar{\omega}_{[\bar{p}\bar{q}\bar{r}]} + \tfrac{1}{6}H_{\bar{p}\bar{q}\bar{r}}\right) , \\
\Phi^{p}{}_{pq} &= \tfrac{1}{\sqrt{2}}\left(e^{p\mu}\omega_{\mu pq} - 2e_{q}{}^{\mu}\partial_{\mu}\phi\right) , &\qquad \bar{\Phi}^{\bar{p}}{}_{\bar{p}\bar{q}} &= \tfrac{1}{\sqrt{2}}\left(\bar{e}^{\bar{p}\mu}\bar{\omega}_{\mu\bar{p}\bar{q}} - 2\bar{e}_{\bar{q}}{}^{\mu}\partial_{\mu}\phi\right) ,
\end{aligned} \tag{2.82}$$

where we have a pair of undoubled spin connections,

$$\omega_{\mu pq} = e_{p}{}^{\nu}(\partial_{\mu}e_{\nu q} - \gamma^{\lambda}_{\mu\nu}e_{\lambda q}) , \qquad \bar{\omega}_{\mu\bar{p}\bar{q}} = \bar{e}_{\bar{p}}{}^{\nu}(\partial_{\mu}\bar{e}_{\nu\bar{q}} - \gamma^{\lambda}_{\mu\nu}\bar{e}_{\lambda\bar{q}}) , \tag{2.83}$$

which along with the Christoffel connection $\gamma^{\lambda}_{\mu\nu}$ imply an undoubled master derivative:

$$\bigtriangledown_{\mu} := \partial_{\mu} + \gamma_{\mu} + \omega_{\mu} + \bar{\omega}_{\mu} , \quad \bigtriangledown_{\mu}e_{\nu}{}^{p} = 0 , \quad \bigtriangledown_{\mu}\eta_{pq} = 0 , \quad \bigtriangledown_{\mu}\bar{e}_{\nu}{}^{\bar{q}} = 0, \quad \bigtriangledown_{\mu}\bar{\eta}_{\bar{p}\bar{q}} = 0 , \quad \bigtriangledown_{\lambda}g_{\mu\nu} = 0 . \tag{2.84}$$

For example, from [67] we obtain for a $\mathbf{Spin}(1,D-1)$ tensor,

$$\mathcal{D}_{\bar{p}}T_{q_1q_2\cdots q_n} = \frac{1}{\sqrt{2}}\bar{e}_{\bar{p}}{}^{\mu}\left[\partial_{\mu}T_{q_1q_2\cdots q_n} + \sum_{i=1}^{n}\left(\omega_{\mu q_i}{}^{r} + \tfrac{1}{2}H_{\mu q_i}{}^{r}\right)T_{q_1\cdots q_{i-1}rq_{i+1}\cdots q_n}\right] , \tag{2.85}$$

and for a weightless spinor, we produce known expressions from generalised geometry [84],

$$\mathcal{D}_{\bar{p}}\rho = \tfrac{1}{\sqrt{2}}\left(\partial_{\bar{p}}\rho + \tfrac{1}{4}\omega_{\bar{p}qr}\gamma^{qr}\rho + \tfrac{1}{8}H_{\bar{p}qr}\gamma^{qr}\rho\right) , \quad \gamma^{p}\mathcal{D}_{p}\rho = \tfrac{1}{\sqrt{2}}\gamma^{\mu}\left(\partial_{\mu}\rho + \tfrac{1}{4}\omega_{\mu qr}\gamma^{qr}\rho + \tfrac{1}{24}H_{\mu qr}\gamma^{qr}\rho - \partial_{\mu}\phi\rho\right) . \tag{2.86}$$

Furthermore, choosing the gauge $e_{\mu}{}^{p} \equiv \bar{e}_{\mu}{}^{\bar{p}}$ reduces the twofold spin groups to a diagonal subgroup, which can be identified with the single spin group of GR. This gauge condition forces the spinors and the RR fields to transform under $\mathbf{O}(D,D)$ rotations [34], and it allows the single RR potential field to be expanded using gamma matrices:

$$\mathcal{C}^{\alpha}{}_{\beta} = \sum_{p}\frac{1}{p!}C_{\mu_1\mu_2\cdots\mu_p}(\gamma^{\mu_1\mu_2\cdots\mu_p})^{\alpha}{}_{\beta} , \tag{2.87}$$

such that the conventional (even or odd) RR form fields appear, and the pair of nilpotent differential operators (2.78) reduce to a twisted exterior derivative and its Hodge dual [78]:

$$\mathcal{D}_{+} \quad\longrightarrow\quad \mathrm{d} + (H - \mathrm{d}\phi)\wedge , \qquad\qquad \mathcal{D}_{-} \quad\longrightarrow\quad \star\left[\,\mathrm{d} + (H - \mathrm{d}\phi)\,\wedge\,\right]\star\ . \tag{2.88}$$

For the non-Riemannian parametrisations of the fully covariant derivatives (2.65) and doubled Yang–Mills theory (2.70), we refer to [85] (section 4.3 therein) and [70], respectively.

2.6 Curvatures

Now we turn to curvatures. The semi-covariant four-index Riemann curvature is defined as [65]:

$$S_{ABCD} := \tfrac{1}{2}\left(\mathfrak{R}_{ABCD} + \mathfrak{R}_{CDAB} - \Gamma^{E}{}_{AB}\Gamma_{ECD}\right) . \tag{2.89}$$

Here Γ_{ABC} is the DFT Christoffel connection (2.41) and $\mathfrak{R}_{ABCD}$ represents its 'field strength':[9]

$$\mathfrak{R}_{CDAB} = \partial_A\Gamma_{BCD} - \partial_B\Gamma_{ACD} + \Gamma_{AC}{}^{E}\Gamma_{BED} - \Gamma_{BC}{}^{E}\Gamma_{AED} , \tag{2.90}$$

which arises in the commutator of the semi-covariant derivatives,

$$[\nabla_A , \nabla_B]\, T_{C_1C_2\cdots C_n} = -\Gamma^{D}{}_{AB}\nabla_D T_{C_1C_2\cdots C_n} + \sum_{i=1}^{n} \mathfrak{R}_{C_i}{}^{D}{}_{AB} T_{C_1\cdots C_{i-1}DC_{i+1}\cdots C_n} , \tag{2.91}$$

satisfies symmetric and projective properties,

$$\mathfrak{R}_{ABCD} = \mathfrak{R}_{[AB][CD]} = (P_A{}^{E}P_B{}^{F} + \bar{P}_A{}^{E}\bar{P}_B{}^{F})\mathfrak{R}_{EFCD} , \tag{2.92}$$

and, under arbitrary variation of the Christoffel symbols (2.56), transforms infinitesimally as

$$\delta\mathfrak{R}_{ABCD} = \nabla_C\delta\Gamma_{DAB} - \nabla_D\delta\Gamma_{CAB} + \Gamma^{E}{}_{CD}\delta\Gamma_{EAB} . \tag{2.93}$$

In particular, under doubled-yet-gauged diffeomorphisms, it varies as

$$\begin{aligned}\delta_\xi\mathfrak{R}_{ABCD} \;=\;& \hat{\mathcal{L}}_\xi\mathfrak{R}_{ABCD} - 2\Gamma^{E}{}_{AB}\partial_E\partial_{[C}\xi_{D]} + 2\Gamma^{E}{}_{CD}(\mathcal{P}{+}\bar{\mathcal{P}})_{EAB}{}^{FGH}\partial_F\partial_{[G}\xi_{H]}\\ &+4\nabla_{[C}\Big((\mathcal{P}{+}\bar{\mathcal{P}})_{D]AB}{}^{FGH}\partial_F\partial_{[G}\xi_{H]}\Big) .\end{aligned} \tag{2.94}$$

Further, the Jacobi identity of the commutators (2.91) implies the following two sets of identities [29]:

$$\mathfrak{R}_{[A}{}^{D}{}_{BC]} + \nabla_{[A}\Gamma^{D}{}_{BC]} + \Gamma^{D}{}_{E[A}\Gamma^{E}{}_{BC]} = 0 , \qquad \nabla_{[A}\mathfrak{R}^{DE}{}_{BC]} + \mathfrak{R}^{DE}{}_{F[A}\Gamma^{F}{}_{BC]} = 0 . \tag{2.95}$$

Consequently, the semi-covariant Riemann curvature (2.89) appears through a projected commutator,

$$[\mathcal{D}_p , \mathcal{D}_{\bar{q}}]\, T_{C_1C_2\cdots C_n} = \sum_{i=1}^{n} 2S_{p\bar{q}C_i}{}^{D} T_{C_1\cdots C_{i-1}DC_{i+1}\cdots C_n} , \tag{2.96}$$

[9]In accordance with the decomposition of an $\mathbf{O}(D,D)$ index into D-dimensional upper and (dual) lower indices, the field strength $\mathfrak{R}_{ABCD}$ includes $\mathfrak{R}^{\kappa}{}_{\lambda\mu\nu}$ which should not be confused with the ordinary (undoubled) Riemann curvature $R^{\kappa}{}_{\lambda\mu\nu}$. Notably, S follows R alphabetically, signifying its intended connection within the semi-covariant formalism.

satisfies symmetric properties, including an algebraic Bianchi identity,

$$S_{ABCD} = S_{CDAB} = S_{[AB][CD]}\,, \qquad\qquad S_{A[BCD]} = 0\,, \tag{2.97}$$

and transforms as total (covariant) derivatives under the arbitrary variation of the Christoffel symbols,

$$\delta S_{ABCD} = \nabla_{[A}\delta\Gamma_{B]CD} + \nabla_{[C}\delta\Gamma_{D]AB}\,. \tag{2.98}$$

In particular, it is semi-covariant under doubled-yet-gauged diffeomorphisms:

$$\delta_\xi S_{ABCD} = \hat{\mathcal{L}}_\xi S_{ABCD} + 2\nabla_{[A}\Big((\mathcal{P}{+}\bar{\mathcal{P}})_{B][CD]}{}^{EFG}\partial_E\partial_F\xi_G\Big) + 2\nabla_{[C}\Big((\mathcal{P}{+}\bar{\mathcal{P}})_{D][AB]}{}^{EFG}\partial_E\partial_F\xi_G\Big)\,. \tag{2.99}$$

Obvious methods of eliminating the anomalous terms via projection end up producing only identically vanishing trivial quantities,

$$S_{pq\bar{p}\bar{q}} = 0\,, \qquad\qquad S_{\bar{p}\bar{q}pq} = 0\,, \qquad\qquad S_{p\bar{p}q\bar{q}} = 0\,, \tag{2.100}$$

and there appears to be no fully covariant four-index curvature in DFT [65, 86]. Fully covariant curvature tensors are then obtained by contracting the indices: with $S_{AB} = S_{BA} = S^C{}_{ACB}$, we have the fully covariant two-index or Ricci curvature in DFT,

$$S_{p\bar{q}} = V^A{}_p\bar{V}^B{}_{\bar{q}}S_{AB}\,, \tag{2.101}$$

and further the scalar curvature,[10]

$$S_{(0)} := \left(P^{AC}P^{BD} - \bar{P}^{AC}\bar{P}^{BD}\right)S_{ABCD} = S_{pq}{}^{pq} - S_{\bar{p}\bar{q}}{}^{\bar{p}\bar{q}}\,. \tag{2.102}$$

These contain both $\mathcal{H}_{AB}$ and d through the Christoffel symbols (2.41): the generalised metric alone cannot generate the covariant curvature [29]. Explicitly, we recover the original expression in [7],

$$S_{(0)} = \mathcal{H}^{AB}\big(\tfrac{1}{8}\partial_A\mathcal{H}_{CD}\partial_B\mathcal{H}^{CD} + \tfrac{1}{2}\partial_C\mathcal{H}_A{}^D\partial_D\mathcal{H}_B{}^C - 4\partial_A d\partial_B d + 4\partial_A\partial_B d\big) - \partial_A\partial_B\mathcal{H}^{AB} + 4\partial_A\mathcal{H}^{AB}\partial_B d\,. \tag{2.103}$$

Like (2.100), the Ricci curvature satisfies the identities [65, 68]:

$$S_{pr\bar{q}}{}^r = S_{p\bar{r}\bar{q}}{}^{\bar{r}} = \tfrac{1}{2}S_{p\bar{q}}\,, \tag{2.104}$$

[10] The subscript (0) indicates the scalar nature of $S_{(0)}$ and serves to distinguish $S_{(0)}$ from the notation used for an action, *e.g.* (2.110).

and it arises through the commutators of the covariant differential operators [68, 84]: using (2.96) and (2.104),

$$\big[\mathcal{D}_p,\mathcal{D}_{\bar{q}}\big]T^p=S_{p\bar{q}}T^p\,,\quad \big[\mathcal{D}_{\bar{q}},\mathcal{D}_p\big]T^{\bar{q}}=S_{p\bar{q}}T^{\bar{q}}\,,\quad \big[\gamma^p\mathcal{D}_p,\mathcal{D}_{\bar{q}}\big]\varepsilon=\tfrac{1}{2}S_{p\bar{q}}\gamma^p\varepsilon\,,\quad \big[\bar{\gamma}^{\bar{q}}\mathcal{D}_{\bar{q}},\mathcal{D}_p\big]\varepsilon'=\tfrac{1}{2}S_{p\bar{q}}\gamma^{\bar{q}}\varepsilon'\,. \tag{2.105}$$

Lastly, the scalar curvature satisfies

$$S_{pq}{}^{pq}+S_{\bar{p}\bar{q}}{}^{\bar{p}\bar{q}}=0\,,\qquad\qquad S_{(0)}=2S_{pq}{}^{pq}=-2S_{\bar{p}\bar{q}}{}^{\bar{p}\bar{q}}\,, \tag{2.106}$$

and manifests itself through successive application of the Dirac operators (2.77),

$$(\gamma^p\mathcal{D}_p)^2\varepsilon+\mathcal{D}_{\bar{p}}\mathcal{D}^{\bar{p}}\varepsilon=-\tfrac{1}{4}S_{pq}{}^{pq}\varepsilon=-\tfrac{1}{8}S_{(0)}\varepsilon\,,\qquad (\bar{\gamma}^{\bar{p}}\mathcal{D}_{\bar{p}})^2\varepsilon'+\mathcal{D}_p\mathcal{D}^p\varepsilon'=-\tfrac{1}{4}S_{\bar{p}\bar{q}}{}^{\bar{p}\bar{q}}\varepsilon'=\tfrac{1}{8}S_{(0)}\varepsilon'\,. \tag{2.107}$$

For the curvatures of the twofold spin connections and their relation to $\mathfrak{R}_{ABCD}$ and S_{ABCD}, we refer to [68] (section 2 therein).

Restricting to the Riemannian parametrisation (2.37), (2.39), we have explicitly,

$$S_{p\bar{q}}=\tfrac{1}{2}e_p{}^{\mu}\bar{e}_{\bar{q}}{}^{\nu}\Big[R_{\mu\nu}+2\nabla_\mu(\partial_\nu\phi)-\tfrac{1}{4}H_{\mu\rho\sigma}H_\nu{}^{\rho\sigma}+\tfrac{1}{2}e^{2\phi}\nabla^\rho\big(e^{-2\phi}H_{\rho\mu\nu}\big)\Big]\,, \tag{2.108}$$

and

$$S_{(0)}=R+4\Box\phi-4\partial_\mu\phi\partial^\mu\phi-\tfrac{1}{12}H_{\lambda\mu\nu}H^{\lambda\mu\nu}\,. \tag{2.109}$$

For non-Riemannian parametrisations, we refer to [85] (section 4.3 therein).

> **Summary.** While a fully covariant four-index curvature is absent, Ricci and scalar curvatures exist and underpin the DFT action.

2.7 Doubled Einstein–Hilbert Action & Gravitational Wave

> **Aim.** To formulate the DFT action and examine its variation, conserved currents, and wave equations.

The 'pure' DFT action, or the doubled Einstein–Hilbert action, is naturally given by the scalar curvature $S_{(0)}$ multiplied by the $\mathbf{O}(D,D)$-symmetric integral measure e^{-2d}, integrated over a section Σ_D:

$$S_{\mathbf{DFT}}=\int_{\Sigma_D}e^{-2d}S_{(0)}\,. \tag{2.110}$$

From the compatibility condition of the semi-covariant derivative (2.46) and the nice properties of the semi-covariant four-index curvature (2.97), (2.98), it is straightforward to vary the pure DFT Lagrangian:

$$\delta\big(e^{-2d}S_{(0)}\big)=4e^{-2d}\big(\bar{V}_{A}{}^{\bar{q}}\delta V^{Ap}S_{p\bar{q}}-\tfrac{1}{2}\delta d\,S_{(0)}\big)+\partial_{A}\big(e^{-2d}\Theta^{A}\big)\,,\tag{2.111}$$

where in the total derivative we have set [31, 87]

$$\Theta^{A}=2\big(P^{AC}P^{BD}-\bar{P}^{AC}\bar{P}^{BD}\big)\delta\Gamma_{BCD}=4\mathcal{H}^{AB}\partial_{B}\delta d-\nabla_{B}\delta\mathcal{H}^{AB}\,.\tag{2.112}$$

It is worthwhile to note different ways of rewriting the variation of the vielbeins in (2.111):

$$\bar{V}_{A}{}^{\bar{q}}\delta V^{Ap}=-V^{Ap}\delta\bar{V}_{A}{}^{\bar{q}}=\tfrac{1}{2}\big(\bar{V}_{A}{}^{\bar{q}}\delta V^{Ap}-V^{Ap}\delta\bar{V}_{A}{}^{\bar{q}}\big)=\tfrac{1}{2}V^{Ap}\bar{V}^{B\bar{q}}\delta\mathcal{H}_{AB}\,.\tag{2.113}$$

In particular, when the variation is generated by the Kosmann derivative (2.80), *i.e.* $\delta_{\xi}=\widetilde{\mathcal{L}}_{\xi}$, we have

$$\begin{array}{ll}\bar{V}_{A}{}^{\bar{q}}\delta_{\xi}V^{Ap}=\bar{V}_{A}{}^{\bar{q}}\widetilde{\mathcal{L}}_{\xi}V^{Ap}=2\mathcal{D}^{[\bar{q}}\xi^{p]}\,, & \delta_{\xi}d=\widetilde{\mathcal{L}}_{\xi}d=\hat{\mathcal{L}}_{\xi}d=-\tfrac{1}{2}\nabla_{A}\xi^{A}\,,\\ \multicolumn{2}{c}{\delta_{\xi}\mathcal{H}_{AB}=\widetilde{\mathcal{L}}_{\xi}\mathcal{H}_{AB}=\hat{\mathcal{L}}_{\xi}\mathcal{H}_{AB}=8\bar{P}_{(A}{}^{C}P_{B)}{}^{D}\nabla_{[C}\xi_{D]}=8\bar{V}_{(A}{}^{\bar{q}}V_{B)}{}^{p}\mathcal{D}_{[\bar{q}}\xi_{p]}\,.}\end{array}\tag{2.114}$$

Applying these to (2.111), *i.e.* the action principle, we can identify the off-shell conserved Einstein curvature in DFT, which satisfies a differential Bianchi identity [31]:

$$G_{AB}=4(PS\bar{P})_{[AB]}-\tfrac{1}{2}\mathcal{J}_{AB}S_{(0)}=4V_{[A}{}^{p}\bar{V}_{B]}{}^{\bar{q}}S_{p\bar{q}}-\tfrac{1}{2}\mathcal{J}_{AB}S_{(0)}\,,\qquad\nabla_{A}G^{AB}=0\,.\tag{2.115}$$

The complete equations of motion of the pure DFT action—derived from the action principle (2.111)—are, *a priori*, expressed by the independent vanishing of the Ricci curvature and the scalar curvature [2, 7]. However, by utilising the relations, $G_{A}{}^{A}=-DS_{(0)}$ and $(PG\bar{P})_{AB}=2(PS\bar{P})_{AB}$, the equations of motion can be seen, in a unified manner, as equivalent to the vanishing of the Einstein curvature [31],

$$S_{p\bar{q}}=0\quad\&\quad S_{(0)}=0\qquad\Longleftrightarrow\qquad G_{AB}=0\,.\tag{2.116}$$

Although the Einstein curvature is not symmetric, $G_{AB}\neq G_{BA}$, and does not satisfy $\nabla_{B}G^{AB}\neq 0$, it is possible to symmetrise the curvature by multiplying the generalised metric from the right. Specifically,

$$(G\mathcal{H})_{AB}=(G\mathcal{H})_{BA}=G_{AC}\mathcal{H}^{C}{}_{B}=-4V_{(A}{}^{p}\bar{V}_{B)}{}^{\bar{q}}S_{p\bar{q}}-\tfrac{1}{2}\mathcal{H}_{AB}S_{(0)}\,,\qquad\nabla_{A}(G\mathcal{H})^{AB}=0\,.\tag{2.117}$$

Under the $(0,0)$ Riemannian parametrisation, from (2.109), the pure DFT action (2.110) coincides with the NSNS gravity action up to a total derivative (denoted by $\simeq$):

$$S_{\mathbf{DFT}} = \int_{\Sigma_D} e^{-2d} S_{(0)} \simeq \int \mathrm{d}^D x \,\sqrt{-g} e^{-2\phi} \Big(R + 4\partial_\mu \phi \partial^\mu \phi - \tfrac{1}{12} H_{\lambda\mu\nu} H^{\lambda\mu\nu} \Big) \,, \tag{2.118}$$

and the upper left $D \times D$ block of $(G\mathcal{H})_{AB}$ contains the undoubled Einstein curvature:

$$(G\mathcal{H})^{\mu\nu} = R^{\mu\nu} - \tfrac{1}{2} g^{\mu\nu} R + 2\nabla^\mu(\partial^\nu \phi) - 2g^{\mu\nu}(\Box\phi - \partial_\sigma\phi\partial^\sigma\phi) - \tfrac{1}{4} H^{\mu\rho\sigma} H^{\nu}{}_{\rho\sigma} + \tfrac{1}{24} g^{\mu\nu} H_{\rho\sigma\tau} H^{\rho\sigma\tau} \,. \tag{2.119}$$

Moreover, the differential Bianchi identity (2.115) decomposes into two separate identities:

$$\nabla_\mu \big(R^{\mu\nu} - \tfrac{1}{2} g^{\mu\nu} R \big) = 0 \,, \qquad\qquad \nabla_\mu \nabla_\nu \big(e^{-2\phi} H^{\lambda\mu\nu} \big) = 0 \,. \tag{2.120}$$

✓ Gamma Squared Action & Noether Currents

Aim. To rewrite the action in Γ^2 form and identify associated symmetries.

The doubled Einstein–Hilbert action (2.110) contains terms with two derivatives of the fundamental fields which can be eliminated by a partial integral. Subtracting a specific total derivative [31],

$$\mathcal{L}_{\Gamma^2} = e^{-2d} S_{(0)} - \partial_A \big(e^{-2d} B^A \big) \,, \qquad B^A = 2(P^{AC} P^{BD} - \bar{P}^{AC} \bar{P}^{BD}) \Gamma_{BCD} = 4\mathcal{H}^{AB} \partial_B d - \partial_B \mathcal{H}^{AB} \,, \tag{2.121}$$

we acquire a doubled Γ^2-action, free of any two-derivative terms (*c.f.* [88], [89]),

$$S_{\Gamma^2} = \int_{\Sigma_D} \mathcal{L}_{\Gamma^2} = \int_{\Sigma_D} e^{-2d} \big(P^{AC} P^{BD} - \bar{P}^{AC} \bar{P}^{BD} \big) \big(\Gamma_{AC}{}^E \Gamma_{BDE} - \Gamma_{AB}{}^E \Gamma_{DCE} + \tfrac{1}{2} \Gamma^E{}_{AB} \Gamma_{ECD} \big) \,, \tag{2.122}$$

which still admits diffeomorphisms as a Noether symmetry. We note generically from (2.111) and (2.121),

$$\delta\mathcal{H}_{BC} \frac{\partial \mathcal{L}_{\Gamma^2}}{\partial(\partial_A \mathcal{H}_{BC})} + \delta d \frac{\partial \mathcal{L}_{\Gamma^2}}{\partial(\partial_A d)} = e^{-2d} \Theta^A - \delta \big(e^{-2d} B^A \big) \,, \tag{2.123}$$

and especially under diffeomorphisms,

$$\delta_\xi \mathcal{L}_{\Gamma^2} = \partial_A \Big[\xi^A e^{-2d} S_{(0)} - \delta_\xi \big(e^{-2d} B^A \big) \Big] \,. \tag{2.124}$$

Thus, with the explicit expression of Θ^A (2.112) and the commutator relations (2.105), the on-shell conserved Noether current for the doubled-yet-gauged diffeomorphisms is [31]

$$J^A_{\mathrm{on-shell}} = e^{-2d} \big(4\mathcal{H}^{AB} \partial_B \delta_\xi d - \nabla_B \delta_\xi \mathcal{H}^{AB} - \xi^A S_{(0)} \big) = \partial_B \big(e^{-2d} K^{[AB]} \big) + 2e^{2d} G^A{}_B \xi^B + \mathrm{j}^A \,, \tag{2.125}$$

where K^{AB} is a skew-symmetric Noether potential, constituting an off-shell conserved Noether current,

$$K^{AB} = 4\bar{V}^{[A}{}_{\bar{p}}V^{B]}{}_{q}\big(\mathcal{D}^{\bar{p}}\xi^{q} + \mathcal{D}^{q}\xi^{\bar{p}}\big)\,, \qquad\qquad J^{A}_{\rm off-shell} = \partial_{B}\big(e^{-2d}K^{[AB]}\big)\,, \tag{2.126}$$

and j^A denotes a harmless derivative-index-valued vector,

$$\mathrm{j}^{A} = 2e^{-2d}\Big[V_{B}{}^{p}\bar{V}_{C}{}^{\bar{q}}\mathcal{D}_{(p}\xi_{\bar{q})}\partial^{A}\mathcal{H}^{BC} - \partial^{A}\big(\mathcal{H}^{BC}\nabla_{B}\xi_{C}\big)\Big]\,, \qquad \mathrm{j}^{A}\partial_{A} = 0\,, \tag{2.127}$$

which does not contribute to any Noether charge.

✓ Box Operator That Unveils The Riemann Curvature Tensor: Gravitational Wave Equations

Aim. To build a covariant box operator that encodes curvature and governs wave equations.

By combining the results from equations such as (2.71), (2.94), and (2.99), it is possible to construct a pair of fully covariant d'Alembertians that act on an arbitrary tensor density $T_{A_1A_2\cdots A_n}$ (2.5) and incorporate S_{ABCD} (2.89) and $\mathfrak{R}_{ABCD}$ (2.90) [90]:

$$\begin{array}{rcl}\boldsymbol{\Delta}T_{A_1A_2\cdots A_n} & := & P^{BC}\nabla_{B}\nabla_{C}T_{A_1A_2\cdots A_n}\\ & & +\displaystyle\sum_{i=1}^{n}\, 2P_{A_i}{}^{C}P_{B}{}^{D}\Big(\mathfrak{R}_{[CD]} - \tfrac{1}{2}\Gamma^{EF}{}_{C}\Gamma_{EFD} - \Gamma^{E}{}_{CD}\nabla_{E}\Big)\,T_{A_1\cdots A_{i-1}}{}^{B}{}_{A_{i+1}\cdots A_n}\\ & & +\displaystyle\sum_{i<j}\, 2\left(\begin{array}{l} P_{A_i}{}^{D}P_{B}{}^{E}\mathfrak{R}_{A_jCDE}\\ +P_{A_j}{}^{D}P_{C}{}^{E}\mathfrak{R}_{A_iBDE}\\ -2P_{A_i}{}^{D}P_{B}{}^{E}P_{A_j}{}^{F}P_{C}{}^{G}S_{DEFG}\end{array}\right)T_{A_1\cdots A_{i-1}}{}^{B}{}_{A_{i+1}\cdots A_{j-1}}{}^{C}{}_{A_{j+1}\cdots A_n}\,,\end{array} \tag{2.128}$$

and with the replacement of every explicit occurrence of the projector $P_A{}^B$ with the opposite projector $\bar{P}_A{}^B$,

$$\begin{aligned}\bar{\boldsymbol{\Delta}}T_{A_1A_2\cdots A_n} \;&:=\; \bar{P}^{BC}\nabla_B\nabla_C T_{A_1A_2\cdots A_n}\\ &+\sum_{i=1}^{n}\,2\bar{P}_{A_i}{}^C\bar{P}_B{}^D\Big(\mathfrak{R}_{[CD]}-\tfrac{1}{2}\Gamma^{EF}{}_C\Gamma_{EFD}-\Gamma^E{}_{CD}\nabla_E\Big)\,T_{A_1\cdots A_{i-1}}{}^B{}_{A_{i+1}\cdots A_n}\\ &+\sum_{i<j}\,2\left(\begin{array}{l}\bar{P}_{A_i}{}^D\bar{P}_B{}^E\mathfrak{R}_{A_jCDE}\\ +\bar{P}_{A_j}{}^D\bar{P}_C{}^E\mathfrak{R}_{A_iBDE}\\ -2\bar{P}_{A_i}{}^D\bar{P}_B{}^E\bar{P}_{A_j}{}^F\bar{P}_C{}^GS_{DEFG}\end{array}\right)T_{A_1\cdots A_{i-1}}{}^B{}_{A_{i+1}\cdots A_{j-1}}{}^C{}_{A_{j+1}\cdots A_n}\,.\end{aligned} \tag{2.129}$$

While these two mirroring operators annihilate each other identically due to the section condition:

$$\big(\boldsymbol{\Delta}+\bar{\boldsymbol{\Delta}}\big)\,T_{A_1A_2\cdots A_n}=0\,, \tag{2.130}$$

their difference defines the fully covariant *box operator*:

$$\Box T_{A_1A_2\cdots A_n}:=\big(\boldsymbol{\Delta}-\bar{\boldsymbol{\Delta}}\big)\,T_{A_1A_2\cdots A_n}=\mathcal{H}^{BC}\nabla_B\nabla_C T_{A_1A_2\cdots A_n}+\sum_i\;\cdots\;+\sum_{i<j}\;\cdots\;. \tag{2.131}$$

In particular, acting on projected tensors such as $\bar{P}_{C_1}{}^{D_1}\cdots\bar{P}_{C_n}{}^{D_n}T_{D_1\cdots D_n}$, $P_{C_1}{}^{D_1}\cdots P_{C_n}{}^{D_n}T_{D_1\cdots D_n}$, and $(PT\bar{P})_{AB}=P_A{}^C\bar{P}_B{}^DT_{CD}$, the pair of d'Alembertians and the box operator produce the two expressions in (2.72) and the following result:

$$\begin{aligned}\Box(PT\bar{P})_{AB} \;&=\; \mathcal{H}^{CD}\nabla_C\nabla_D(PT\bar{P})_{AB}-2P_A{}^C\bar{P}_B{}^D(\mathfrak{R}_{CEDF}-\mathfrak{R}_{DFCE})(PT\bar{P})^{EF}\\ &+2P_A{}^C(\mathfrak{R}_{[CD]}-\tfrac{1}{2}\Gamma^{EF}{}_C\Gamma_{EFD}-\Gamma^E{}_{CD}\nabla_E)(PT\bar{P})^D{}_B\\ &-2\bar{P}_B{}^C(\mathfrak{R}_{[CD]}-\tfrac{1}{2}\Gamma^{EF}{}_C\Gamma_{EFD}-\Gamma^E{}_{CD}\nabla_E)(PT\bar{P})_A{}^D\,.\end{aligned} \tag{2.132}$$

Equivalent yet more compact expressions are available through contraction with the vielbeins, $V_{Ap}, \bar{V}_{B\bar{q}}$:

$$\begin{aligned}\boldsymbol{\Delta}T_{p\bar{q}}&=\mathcal{D}_r\mathcal{D}^rT_{p\bar{q}}+2\mathfrak{R}_{\bar{q}\bar{s}pr}T^{r\bar{s}}+2\big(\mathfrak{R}_{[pr]}-\tfrac{1}{2}\Gamma^{AB}{}_p\Gamma_{ABr}-\Gamma^C{}_{pr}\mathcal{D}_C\big)T^r{}_{\bar{q}}\,,\\ \bar{\boldsymbol{\Delta}}T_{p\bar{q}}&=\mathcal{D}_{\bar{r}}\mathcal{D}^{\bar{r}}T_{p\bar{q}}+2\mathfrak{R}_{pr\bar{q}\bar{s}}T^{r\bar{s}}+2\big(\mathfrak{R}_{[\bar{q}\bar{s}]}-\tfrac{1}{2}\Gamma^{AB}{}_{\bar{q}}\Gamma_{AB\bar{s}}-\Gamma^C{}_{\bar{q}\bar{s}}\mathcal{D}_C\big)T_p{}^{\bar{s}}\,.\end{aligned} \tag{2.133}$$

Under the $(0,0)$ Riemannian parametrisation (2.37), (2.39), we set for the two-index tensor,

$$T_{p\bar{q}} = V^{A}{}_{p}\bar{V}^{B}{}_{\bar{q}}T_{AB} = \mathfrak{T}^{\mu\nu}e_{\mu p}\,\bar{e}_{\nu\bar{q}} \qquad \Longleftrightarrow \qquad \mathfrak{T}_{\mu\nu} = -e_{\mu}{}^{p}\bar{e}_{\nu}{}^{\bar{q}}T_{p\bar{q}}\,, \tag{2.134}$$

and the box operator reduces to

$$\hat{\Box}\mathfrak{T}_{\mu\nu} = -e_{\mu}{}^{p}\bar{e}_{\nu}{}^{\bar{q}}\Box T_{p\bar{q}} = e^{2\phi}\bigtriangledown^{\rho}\big(e^{-2\phi}\bigtriangledown_{\rho}\mathfrak{T}_{\mu\nu}\big) - H_{\rho\sigma\mu}\bigtriangledown^{\rho}\mathfrak{T}^{\sigma}{}_{\nu} + H_{\rho\sigma\nu}\bigtriangledown^{\rho}\mathfrak{T}_{\mu}{}^{\sigma} + 2\hat{R}_{\mu}{}^{\rho}{}_{\nu}{}^{\sigma}\mathfrak{T}_{\rho\sigma}\,, \tag{2.135}$$

where $\hat{R}_{\mu}{}^{\rho}{}_{\nu}{}^{\sigma}$ contains the Riemann curvature tensor and the H-flux,

$$\begin{array}{lll}\hat{R}_{\mu}{}^{\rho}{}_{\nu}{}^{\sigma} & = & R_{\mu}{}^{\rho}{}_{\nu}{}^{\sigma} - \frac{1}{2}H_{(\mu}{}^{\rho\kappa}H_{\nu)}{}^{\sigma}{}_{\kappa} - \frac{1}{4}H_{\mu\nu\kappa}H^{\rho\sigma\kappa} + \frac{1}{2}\bigtriangledown_{(\mu}H_{\nu)}{}^{\rho\sigma} + \frac{1}{2}\bigtriangledown^{(\rho}H^{\sigma)}{}_{\mu\nu} \\ & & +\frac{1}{4}\delta_{\mu}{}^{\rho}\Big[e^{2\phi}\bigtriangledown_{\kappa}\big(e^{-2\phi}H^{\kappa\sigma}{}_{\nu}\big) - \frac{1}{2}H_{\nu\kappa\lambda}H^{\sigma\kappa\lambda}\Big] - \frac{1}{4}\Big[e^{2\phi}\bigtriangledown_{\kappa}\big(e^{-2\phi}H^{\kappa\rho}{}_{\mu}\big) + \frac{1}{2}H_{\mu\kappa\lambda}H^{\rho\kappa\lambda}\Big]\delta_{\nu}{}^{\sigma}\,.\end{array} \tag{2.136}$$

In the context of applications, the imposition of an $\mathbf{O}(D,D)$-symmetric harmonic gauge,

$$e^{2d}\nabla_{A}\delta\big(e^{-2d}\mathcal{H}^{AB}\big) = \nabla_{A}\delta\mathcal{H}^{AB} - 2\mathcal{H}^{AB}\nabla_{A}\delta d = 0\,, \tag{2.137}$$

leads to a simplification of the linearised equations of motion in pure DFT [50] (see also [91, 92]). The resulting $\mathbf{O}(D,D)$-symmetric gravitational wave equations are, with the box operator (2.132) [90]:

$$\Box(P\delta\mathcal{H}\bar{P})_{AB} = 0\,, \qquad\qquad \mathcal{H}^{AB}\nabla_{A}\partial_{B}\delta d = 0\,. \tag{2.138}$$

When applied to a Riemannian background, the $\mathbf{O}(D,D)$-symmetric harmonic gauge condition (2.137) decomposes into the following components:

$$e^{2\phi}\bigtriangledown^{\rho}\big(e^{-2\phi}\delta g_{\rho\mu}\big) - \tfrac{1}{2}H_{\mu}{}^{\rho\sigma}\delta B_{\rho\sigma} + 2\partial_{\mu}\delta d = 0\,, \qquad\qquad e^{2\phi}\bigtriangledown^{\rho}\big(e^{-2\phi}\delta B_{\rho\mu}\big) = 0\,, \tag{2.139}$$

where $\delta d = \delta\phi - \frac{1}{4}g^{\rho\sigma}\delta g_{\rho\sigma}$, and the wave equations (2.138) reduce further, with (2.135) and (2.136), to

$$\hat{\Box}(\delta g_{\mu\nu} - \delta B_{\mu\nu}) = 0\,, \qquad\qquad e^{2\phi}\bigtriangledown^{\rho}\big(e^{-2\phi}\partial_{\rho}\delta d\big) = 0\,. \tag{2.140}$$

2.8 Einstein Double Field Equation: Unified Field Equation

Aim. To couple matter fields to DFT consistently and derive the unified field equation.

By employing the fully covariant derivatives (2.65), (2.66), (2.73), (2.77), and (2.79), it is possible to couple the pure DFT action (2.110) to various forms of matter,

$$S_{\mathbf{DFT-Matter}} = \int_{\Sigma_D} e^{-2d} S_{(0)} \,+\, \mathcal{L}_{\mathbf{Matter}}(\Psi, \mathcal{D}\Psi; d, V, \bar{V})\,, \tag{2.141}$$

where $\mathcal{L}_{\mathbf{Matter}}(\Psi, \mathcal{D}\Psi; d, V, \bar{V})$ represents a Lagrangian density of unit weight for generic matter fields Ψ which are minimally coupled to the gravitational sector $\{d, V_{Ap}, \bar{V}_{A\bar{p}}\}$.

Generalising the case of pure DFT (2.111), while ignoring any surface integral and assuming that $\mathcal{L}_{\mathbf{Matter}}$ is local Lorentz invariant (2.25), the infinitesimal variation of the full action (2.141) is given by [25]:

$$\delta S_{\mathbf{DFT-Matter}} = \int_{\Sigma_D} e^{-2d}\Big[4\bar{V}_A{}^{\bar{q}}\delta V^{Ap}\big(S_{p\bar{q}} - K_{p\bar{q}}\big) - 2\delta d\big(S_{(0)} - T_{(0)}\big)\Big] + \delta\Psi\frac{\delta\mathcal{L}_{\mathbf{Matter}}}{\delta\Psi}\,. \tag{2.142}$$

Here $\frac{\delta\mathcal{L}_{\mathbf{Matter}}}{\delta\Psi}$ denotes the Euler–Lagrange equations for the matter field. Naturally, we are led to define

$$K_{p\bar{q}} = \frac{1}{4}e^{2d}\left(V_{Ap}\frac{\delta\mathcal{L}_{\mathbf{Matter}}}{\delta\bar{V}_A{}^{\bar{q}}} - \bar{V}_{A\bar{q}}\frac{\delta\mathcal{L}_{\mathbf{Matter}}}{\delta V_A{}^p}\right)\,, \qquad\qquad T_{(0)} = \frac{1}{2}e^{2d}\,\frac{\delta\mathcal{L}_{\mathbf{Matter}}}{\delta d}\,, \tag{2.143}$$

which constitute the on-shell conserved, $\mathbf{O}(D,D)$-symmetric energy-momentum tensor in DFT:

$$T_{AB} = 4V_{[A}{}^p\bar{V}_{B]}{}^{\bar{q}}K_{p\bar{q}} - \tfrac{1}{2}\mathcal{J}_{AB}T_{(0)}\,, \qquad\qquad \nabla_A T^{AB} = 0\,, \tag{2.144}$$

where the conservation condition requires the Euler–Lagrange equations for the matter field. Notably, when matter couples directly to the generalised metric rather than the vielbeins, we simply have

$$K_{p\bar{q}} = -e^{2d}\frac{\delta\mathcal{L}_{\mathbf{Matter}}}{\delta\mathcal{H}_{AB}}V_{Ap}\bar{V}_{B\bar{q}}\,. \tag{2.145}$$

The unified field equation of DFT is obtained by equating the Einstein curvature (2.115) with the energy-momentum tensor (2.144):

$$G_{AB} = T_{AB}\,, \tag{2.146}$$

which was dubbed the Einstein Double Field Equation (EDFE) [25].

On a $(0,0)$ Riemannian background, the energy-momentum tensor can be parametrised as

$$T_{AB} = \begin{pmatrix} -K^{[\mu\nu]} & K^{(\mu\lambda)}g_{\lambda\sigma} + K^{[\mu\lambda]}B_{\lambda\sigma} - \tfrac{1}{2}\delta^\mu{}_\sigma T_{(0)} \\ -g_{\rho\kappa}K^{(\kappa\nu)} - B_{\rho\kappa}K^{[\kappa\nu]} - \tfrac{1}{2}\delta_\rho{}^\nu T_{(0)} & K_{[\rho\sigma]} + B_{\rho\kappa}K^{[\kappa\lambda]}B_{\lambda\sigma} + B_\rho{}^\kappa K_{(\kappa\sigma)} + K_{(\rho\lambda)}B^\lambda{}_\sigma \end{pmatrix}\,, \tag{2.147}$$

and the EDFE yields the following three sets of equations:

$$\begin{array}{rcl} R_{\mu\nu}+2\bigtriangledown_{\mu}(\partial_{\nu}\phi)-\frac{1}{4}H_{\mu\rho\sigma}H_{\nu}{}^{\rho\sigma} & = & K_{(\mu\nu)}\,, \\ \frac{1}{2}e^{2\phi}\bigtriangledown^{\rho}\big(e^{-2\phi}H_{\rho\mu\nu}\big) & = & K_{[\mu\nu]}\,, \\ R+4\Box\phi-4\partial_{\mu}\phi\partial^{\mu}\phi-\frac{1}{12}H_{\lambda\mu\nu}H^{\lambda\mu\nu} & = & T_{(0)}\,. \end{array} \tag{2.148}$$

Each equation can be understood as the equation of motion of the full action $S_{\mathbf{DFT-Matter}}$ for $g_{\mu\nu}$, $B_{\mu\nu}$, and d, respectively (rather than g, B, ϕ). In view of $d=\phi-\frac{1}{2}\ln\sqrt{-g}$ (2.39), the traditional energy-momentum tensor in GR is given by $K_{(\mu\nu)}$ and $T_{(0)}$ [93]:

$$T^{\mathbf{GR}}_{\mu\nu}=e^{-2\phi}\left[K_{(\mu\nu)}-\frac{1}{2}g_{\mu\nu}T_{(0)}\right]\,. \tag{2.149}$$

The on-shell conservation (2.144) reduces to

$$\bigtriangledown^{\mu}K_{(\mu\nu)}-2\partial^{\mu}\phi\,K_{(\mu\nu)}+\tfrac{1}{2}H_{\nu}{}^{\lambda\mu}K_{[\lambda\mu]}-\tfrac{1}{2}\partial_{\nu}T_{(0)}=0\,,\qquad\qquad \bigtriangledown_{\mu}\big(e^{-2\phi}K^{[\mu\nu]}\big)=0\,, \tag{2.150}$$

which are consistent with the geometric counterparts (2.120). In particular, the middle equation in (2.148) and the latter in (2.150) represent the stringy two-form generalisations of Maxwell's equations (1.2) and the corresponding conservation laws: $\bigtriangledown_{\lambda}F^{\lambda\mu}=J^{\mu}$ and $\bigtriangledown_{\mu}J^{\mu}=0$. Strings, unlike point particles, couple to the two-form B-field rather than the one-form Maxwell vector potential.

It is worth noting that, after subtracting the third equation from the trace of the first in (2.148), we can derive a Klein–Gordon-type equation [94]:

$$\Box\big(e^{-2\phi}\big)=\big(K_{\mu}{}^{\mu}-T_{(0)}+\tfrac{1}{6}H_{\lambda\mu\nu}H^{\lambda\mu\nu}\big)e^{-2\phi}\,, \tag{2.151}$$

where the quantity inside the parentheses on the right-hand side of the equality can be interpreted as the effective or "Chameleon mass" [95, 96] of the dilaton scalar field, $e^{-2\phi}$. This interpretation links the dilaton dynamics to a scalar field with an effective mass that varies based on other fields and their interactions, similar to scalar-tensor theories where the scalar field's mass adjusts to its environment.

✓ $\mathbf{O}(D,D)$ Tells Matter How to Couple to DFT: Equivalence Principle Holds in String Frame

The EDFE (2.146) provides an $\mathbf{O}(D,D)$-symmetric extension of Wheeler's famous insight into gravity: "*Matter tells spacetime how to curve*." The complementary notion, "*Spacetime tells matter how to move*," is realised through the $\mathbf{O}(D,D)$-symmetric minimal coupling of the gravitational fields $\{V_{Ap}, \bar{V}_{B\bar{q}}, d\}$ to matter, as demonstrated in (2.14), (2.15), (2.70), (2.85), (2.86), and (2.141). Consequently, the coupling of the Riemannian trio $\{g_{\mu\nu}, B_{\mu\nu}, \phi\}$ to the Standard Model of particle physics is fully dictated by the $\mathbf{O}(D,D)$ symmetry principle, ensuring the emergence of a fixed structure that would not otherwise arise [35]. Schematically, from (2.70) and (2.86), the action for a scalar field Φ, electromagnetic fields $A_\lambda, F_{\mu\nu}$, and a spinorial fermion ψ with diffeomorphic weight $\omega = \frac{1}{2}$ is given by:

$$\int_{\Sigma_D} \sqrt{-g}e^{-2\phi}\left(-g^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi - \frac{1}{4\alpha}F_{\mu\nu}F^{\mu\nu}\right) + \bar{\psi}\gamma^\lambda\left(\bigtriangledown_\lambda\psi - iA_\lambda\psi + \frac{1}{24}H_{\lambda\mu\nu}\gamma^{\mu\nu}\psi\right). \qquad (2.152)$$

This indicates that bosons couple to the string dilaton, while fermions couple to the H-flux. As a result, they contribute distinct components to the energy-momentum tensor: bosons generate $T_{(0)}$, and fermions generate $K_{[\mu\nu]}$, alongside the shared components $K_{(\mu\nu)}$. With additional components, the gravitational physics in DFT is inherently richer than in General Relativity: D^2+1 *vs.* $\frac{1}{2}D(D+1)$ (off-shell) degrees of freedom.

However, after integrating out the auxiliary potential $\mathfrak{a}^A$, the $\mathbf{O}(D,D)$-symmetric doubled particle action (2.14) reduces, on a Riemannian background (2.33), to the standard (undoubled) relativistic point-particle action minimally coupled solely to the string frame metric $g_{\mu\nu}$. Since the $\mathbf{O}(D,D)$ singlet dilaton d, or e^{-2d}, carries a nontrivial diffeomorphism weight, it cannot couple to the diffeomorphism-invariant doubled particle action (2.14). Consequently, the free-falling motion of particles along geodesics—free from any fifth force—preserves the equivalence principle, not in Einstein frame but in string frame [42]. This result aligns with string theory's foundational premise that matter consists of vibrating tiny strings, which naturally couple to the string frame metric.

Importantly, in the string frame, the kinetic term of the string dilaton has an opposite sign, with the square of its time derivative carrying a negative coefficient (2.118). This property enables the formation of wormholes and drives the accelerated expansion of the Universe, as discussed below.

Summary. The equivalence principle holds in string frame: bosons couple to the dilaton, while fermions couple to the H-flux, producing dynamics beyond GR.

3 Solutions

With the theoretical foundations established, we now turn to specific solutions within the DFT framework.

Any known $D = 10$ IIA or IIB supergravity solution consistently satisfies the equations of motion, including the Einstein Double Field Equation (EDFE), of type II supersymmetric DFT [34]. However, here we focus on more elementary $D = 4$ solutions.

In General Relativity, the Schwarzschild geometry and de Sitter space are fundamental solutions. The Schwarzschild solution describes the spacetime around a black hole or the exterior geometry of a spherical object, while de Sitter space serves as a model for the large-scale structure of the Universe. Below, we present their counterparts within the framework of Double Field Theory (DFT).

3.1 Spherically Symmetric Solution: Generalisation of the Schwarzschild Geometry

Aim. To solve the EDFE in four dimensions under spherical symmetry.

The DFT counterpart to the Schwarzschild geometry in GR is a three-parameter family of vacuum solutions to the $D = 4$ EDFE, *i.e.* $G_{AB} = 0$ or (1.3). These solutions can be traced back to the work of Burgess, Myers, and Quevedo in 1994 [97], who obtained them by performing $\mathbf{SL}(2, \mathbb{R})$ S-duality rotations on a dilaton–metric solution. The solutions were later re-derived [42] as the most general, spherically symmetric, static, asymptotically flat, vacuum geometry of $D = 4$ DFT. We present the solutions with three constants $\{h, a, b\}$ in an isotropic coordinate system [94], where the metric takes the form, with $r = \sqrt{\vec{x}\cdot\vec{x}}$,

$$\mathrm{d}s^2 = g_{tt}(r)\,\mathrm{d}t^2 + g_{rr}(r)\,\mathrm{d}\vec{x}\cdot\mathrm{d}\vec{x}\,. \tag{3.1}$$

The two significant components of the metric are

$$g_{tt}(r) = -e^{2\phi(r)}\left(\tfrac{4r-\sqrt{a^2+b^2}}{4r+\sqrt{a^2+b^2}}\right)^{\frac{2a}{\sqrt{a^2+b^2}}}\,, \quad g_{rr}(r) = e^{2\phi(r)}\left(\tfrac{4r+\sqrt{a^2+b^2}}{4r-\sqrt{a^2+b^2}}\right)^{\frac{2a}{\sqrt{a^2+b^2}}}\left(1-\tfrac{a^2+b^2}{16r^2}\right)^2\,, \tag{3.2}$$

the string dilaton is given, with $\gamma_\pm = \frac{1}{2} \pm \frac{1}{2}\sqrt{1 - h^2/b^2}$, by

$$e^{2\phi} = \gamma_+\left(\frac{4r-\sqrt{a^2+b^2}}{4r+\sqrt{a^2+b^2}}\right)^{\frac{2b}{\sqrt{a^2+b^2}}} + \gamma_-\left(\frac{4r+\sqrt{a^2+b^2}}{4r-\sqrt{a^2+b^2}}\right)^{\frac{2b}{\sqrt{a^2+b^2}}}\,, \tag{3.3}$$

and the B-field features electric H-flux,

$$H_{(3)} = h\sin\vartheta\,\mathrm{d}t\wedge\mathrm{d}\vartheta\wedge\mathrm{d}\varphi = h\,\mathrm{d}t\wedge\left(\frac{\epsilon_{ijk}x^i\,\mathrm{d}x^j\wedge\mathrm{d}x^k}{2r^3}\right)\,. \tag{3.4}$$

Several observations can be made about these solutions.

i) If $b = h = 0$, with $h/b \to 0$, the solution reduces to the Schwarzschild geometry.

ii) If $a = 0$, the solution reduces to a wormhole geometry [98],

$$\mathrm{d}s^2 = \frac{-\mathrm{d}t^2 + \mathrm{d}y^2}{\mathcal{F}(y)} + \mathcal{R}(y)^2\left(\mathrm{d}\vartheta^2 + \sin^2\vartheta\,\mathrm{d}\varphi^2\right), \qquad e^{2\phi(y)} = \frac{1}{|\mathcal{F}(y)|}, \tag{3.5}$$

where

$$\mathcal{R}(y) = \sqrt{y^2 + \tfrac{1}{4}h^2}, \qquad \mathcal{F}(y) = \frac{(y-b_-)(y-b_+)}{y^2 + \frac{1}{4}h^2}, \qquad b_+ = -b\gamma_+, \qquad b_- = b\gamma_-. \tag{3.6}$$

While the wormhole throat is at $y = 0$, the points of $y = b_\pm$ are curvature-wise singular within Riemannian geometry: $R \propto 1/(y - b_\pm)$ as $y \to b_\pm$. However, as a vacuum solution to the EDFE, the geometry sets both the DFT scalar and Ricci curvature trivial, $S_{(0)} = 0$ and $S_{p\bar{q}} = 0$. In fact, by choosing the B-field of the electric H-flux (3.4) appropriately to include a term that is pure gauge,

$$B_{(2)} = h\cos\vartheta\,\mathrm{d}t \wedge \mathrm{d}\varphi + \frac{\mathrm{d}t \wedge \mathrm{d}y}{\mathcal{F}(y)}, \tag{3.7}$$

the DFT metric (2.33) and dilaton (2.39) can both be made entirely non-singular, as can be seen from

$$g^{-1} = \begin{pmatrix} -\mathcal{F} & 0 & 0 & 0 \\ 0 & \mathcal{F} & 0 & 0 \\ 0 & 0 & \frac{1}{\mathcal{R}^2} & 0 \\ 0 & 0 & 0 & \frac{1}{\mathcal{R}^2\sin^2\vartheta} \end{pmatrix}, \quad Bg^{-1} = \left(-g^{-1}B\right)^T = \begin{pmatrix} 0 & 1 & 0 & \frac{h\cos\vartheta}{\mathcal{R}^2\sin^2\vartheta} \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ h\cos\vartheta\mathcal{F} & 0 & 0 & 0 \end{pmatrix}, \tag{3.8}$$

and

$$g - Bg^{-1}B = \begin{pmatrix} \frac{h^2\cos^2\vartheta}{\mathcal{R}^2\sin^2\vartheta} & 0 & 0 & 0 \\ 0 & 0 & 0 & -h\cos\vartheta \\ 0 & 0 & \mathcal{R}^2 & 0 \\ 0 & -h\cos\vartheta & 0 & \mathcal{R}^2\sin^2\vartheta - h^2\mathcal{F}\cos^2\vartheta \end{pmatrix}, \qquad e^{-2d} = \mathcal{R}^2\sin\vartheta. \tag{3.9}$$

Only positive powers of $\mathcal{F}(y)$ appear in (3.8) and (3.9) after cancellation of the negative powers. This sufficiently implies that, as the B-field gauge transformation is a part of doubled-yet-gauged diffeomorphisms, the curvature singularity as characterised within Riemannian geometry is to be identified as a coordinate singularity within DFT [64]. When $\mathcal{F}(y)$ vanishes at $y = b_{\pm}$, the upper-left block of the generalised metric, corresponding to g^{-1}, becomes degenerate, no invertible Riemannian metric is defined, and thus the geometry is non-Riemannian. From the perspective of DFT, the geometry is regular everywhere [64]: it is non-Riemannian at $y = b_{\pm}$ and Riemannian elsewhere. In fact, it is the same type of non-Riemannian geometry, $(n, \bar{n}) = (1, 1)$, as non-relativistic string theory [51], as can be seen from (3.8) and (3.9) [44, 50].

iii) Matching with the parametrised Post-Newtonian (PPN) formalism [99, 100],

$$\mathrm{d}s^2 = -\left(1 - \frac{2MG}{r} + \frac{2\beta_{\mathrm{PPN}}(MG)^2}{r^2} + \cdots\right)\mathrm{d}t^2 + \left(1 + \frac{2\gamma_{\mathrm{PPN}}MG}{r} + \cdots\right)\mathrm{d}x^i\mathrm{d}x^j\delta_{ij}\,, \quad (3.10)$$

we can identify the Newtonian mass and the Eddington–Robertson–Schiff parameters [94],

$$2MG = a + b\sqrt{1 - h^2/b^2}\,, \qquad \beta_{\mathrm{PPN}} = 1 + \left(\frac{h}{a+b\sqrt{1-h^2/b^2}}\right)^2, \qquad \gamma_{\mathrm{PPN}} = 1 - \frac{2b\sqrt{1-h^2/b^2}}{a+b\sqrt{1-h^2/b^2}}\,. \quad (3.11)$$

iv) The vacuum solution (3.1) represents the external geometry of a compact spherical object, such as a star. The full EDFE (2.148) then gives an integral formula for the Newtonian mass [94],

$$MG = \frac{1}{4\pi}\int \mathrm{d}^3x\, e^{-2d}\left(-K_t{}^t - \frac{1}{2}H_{t\vartheta\varphi}H^{t\vartheta\varphi}\right)\,, \quad (3.12)$$

where the integration occurs within the star's interior. Assuming the star's geometry is regular, the integral of the squared electric H-flux term in (3.12) diverges unless $h = 0$. To ensure physical consistency, we impose a weak energy condition, $-K_t{}^t \geq 0$, and assume a finite Newtonian mass, *i.e.* $MG \ll \infty$. Under these conditions, we deduce that the electric H-flux must be trivial: $h = 0$. Consequently, from (3.11), this leads to $\beta_{\mathrm{PPN}} = 1$, which aligns with the result in GR, or correspondingly, the Schwarzschild geometry. The constants a and b are then determined as follows:

$$\begin{aligned} a &= \frac{1}{4\pi}\int \mathrm{d}^3x\, e^{-2d}\left(K_\mu{}^\mu - 2K_t{}^t - T_{(0)} + H_{r\vartheta\varphi}H^{r\vartheta\varphi}\right)\,, \\ b &= \frac{1}{4\pi}\int \mathrm{d}^3x\, e^{-2d}\left(-K_\mu{}^\mu + T_{(0)} - H_{r\vartheta\varphi}H^{r\vartheta\varphi}\right)\,, \end{aligned} \quad (3.13)$$

with their square sum satisfying a nontrivial relation: given the star's radius $r_\star$,

$$\frac{a^2+b^2}{16}=\int_0^{r_\star}\mathrm{d}r\,r\int_r^{r_\star}\frac{\mathrm{d}r'}{r'}\left(\frac{e^{-2d}}{\sin\vartheta}\right)\left(K_r{}^r+K_\vartheta{}^\vartheta-T_{(0)}+\frac{1}{2}H_{r\vartheta\varphi}H^{r\vartheta\varphi}\right)\,. \tag{3.14}$$

From (3.11), $\gamma_{\rm PPN}$ is thereby determined as:

$$\gamma_{\rm PPN}=1+\frac{\displaystyle\int\mathrm{d}^3x\;e^{-2d}\big(K_\mu{}^\mu-T_{(0)}+H_{r\vartheta\varphi}H^{r\vartheta\varphi}\big)}{\displaystyle\int\mathrm{d}^3x\;e^{-2d}\big(-K_t{}^t\big)}\,, \tag{3.15}$$

where the numerator corresponds to the Chameleon mass of the string dilaton (2.151) and the denominator to the Newtonian mass of the star (3.12).

v) The current stringent observational bounds on gravity in the solar system are given by $\gamma_{\rm PPN}=1+(2.1\pm 2.3)\times 10^{-5}$, as determined from the Shapiro time-delay measurements by the Cassini spacecraft [100, 101]. Consequently, DFT can successfully pass solar system tests, provided the dilatonic Chameleon mass is at least 10^{-5} times smaller than the Sun's Newtonian mass.

Summary. The three-parameter family of spherically symmetric solutions to the EDFE in $D=4$ includes Schwarzschild, wormhole, and dilatonic solutions. Post-Newtonian analysis shows consistency with solar-system tests provided the dilaton's Chameleon mass is suppressed.

3.2 Cosmological Solution of Open Universe: Alternative to de Sitter Universe

Aim. To obtain a cosmological solution of DFT consistent with real data.

The de Sitter Universe, while serving as a natural cosmological solution in GR and providing a simple model for the Universe's accelerating expansion, encounters a fundamental limitation in DFT: the cosmological term $\sqrt{-g}\Lambda$ lacks $\mathbf{O}(D,D)$ symmetry, as originally pointed out by Gasperini and Veneziano in 1991 [102]. Consequently, the de Sitter Universe is incompatible with the EDFE (2.146) and the implied $\mathbf{O}(D,D)$-symmetric extension of the Friedmann equations [103]. This incompatibility aligns with the de Sitter swampland conjectures [104–107].

We introduce an alternative cosmological model: an open Universe characterised by negative spatial curvature ($k<0$) that serves as a vacuum solution to the EDFE, providing a compelling alternative to the de Sitter Universe. This solution traces back to the work of Copeland, Lahiri, and Wands in 1994 [108], who derived homogeneous and isotropic solutions to the three beta-function equations on the string worldsheet (1.3). Building upon this foundation, the model was further refined in [93], incorporating essential physical parameters: the Hubble constant H_0, the spatial curvature scale $l = 1/\sqrt{-k}$, the magnetic H-flux $\mathfrak{h}$, as well as additional redundant parameters a_0 and ϕ_0, defined at the (present) conformal time $\eta = \eta_0$.

The vacuum geometry of the open Universe is characterized by the following triplet: the string dilaton,

$$e^{2\phi(\eta)-2\phi_0} = \tfrac{1}{2}\left[1+\sigma\sqrt{1-\tfrac{(\mathfrak{h}l\sinh\zeta)^2}{12a_0^4}}\right]\left[\frac{\tanh\left(\frac{\eta-\eta_0}{l}+\frac{\zeta}{2}\right)}{\tanh\frac{\zeta}{2}}\right]^{\sqrt{3}} + \tfrac{1}{2}\left[1-\sigma\sqrt{1-\tfrac{(\mathfrak{h}l\sinh\zeta)^2}{12a_0^4}}\right]\left[\frac{\tanh\left(\frac{\eta-\eta_0}{l}+\frac{\zeta}{2}\right)}{\tanh\frac{\zeta}{2}}\right]^{-\sqrt{3}}, \tag{3.16}$$

the (homogeneous & isotropic) magnetic H-flux,

$$H_{(3)} = \frac{\mathfrak{h}\, r^2 \sin\vartheta\, \mathrm{d}r\wedge\mathrm{d}\vartheta\wedge\mathrm{d}\varphi}{\sqrt{1+r^2/l^2}}\,, \tag{3.17}$$

and the Friedmann–Lemaître–Robertson–Walker metric with $k = -1/l^2 < 0\,$,

$$\mathrm{d}s^2 = a^2(\eta)\left[-\mathrm{d}\eta^2 + \frac{\mathrm{d}r^2}{1+r^2/l^2} + r^2\mathrm{d}\vartheta^2 + r^2\sin^2\vartheta\,\mathrm{d}\varphi^2\right]\,, \tag{3.18}$$

of which the scale factor is given by the following expression:

$$a^2(\eta) = a_0^2\, e^{2\phi(\eta)-2\phi_0}\, \frac{\sinh\left(2(\eta-\eta_0)/l+\zeta\right)}{\sinh\zeta}\,. \tag{3.19}$$

Here, ζ is a constant and σ is a sign factor ($\sigma^2 = 1$), both determined by the parameters $\{H_0, l, \mathfrak{h}\}$. The Hubble parameter and two density parameters are defined at arbitrary conformal time η as

$$H = \frac{1}{a^2}\frac{\mathrm{d}a}{\mathrm{d}\eta}\,, \qquad \Omega_k = \frac{1}{l^2a^2H^2}\,, \qquad \Omega_{\mathfrak{h}} = \frac{\mathfrak{h}^2}{12a^6H^2}\,. \tag{3.20}$$

At the specific conformal time $\eta = \eta_0$, the analytic solution for the scale factor (3.19) yields the Hubble constant, $H_0 = H(\eta_0)$, as:

$$H_0 = \frac{1}{2a_0 l\sinh\zeta}\left[2\cosh\zeta + \sigma\sqrt{12 - a_0^{-4}\left(\mathfrak{h}l\sinh\zeta\right)^2}\right]\,, \tag{3.21}$$

which can be solved to express ζ and σ in terms of the density parameters at $\eta = \eta_0$. These expressions are

$$\sinh\zeta = \sqrt{\frac{2\Omega_{0,k}}{2+\Omega_{0,k}+3\Omega_{0,\mathfrak{h}}-\sqrt{3+6\Omega_{0,k}+6\Omega_{0,\mathfrak{h}}}}}\,, \qquad \sigma = \frac{\sqrt{3}-\sqrt{1+2\Omega_{0,k}+2\Omega_{0,\mathfrak{h}}}}{\left|\sqrt{3}-\sqrt{1+2\Omega_{0,k}+2\Omega_{0,\mathfrak{h}}}\right|}\,. \tag{3.22}$$

In other words, $\sigma = -1$ if $\Omega_{0,k} + \Omega_{0,\mathfrak{h}} > 1$; otherwise, $\sigma = +1$.

Some comments are in order, as discussed in [93].

i) The deceleration parameter can be expressed in terms of the density parameters as follows:

$$q = -\frac{1}{H^2 a}\left(\frac{\mathrm{d}}{a\mathrm{d}\eta}\right)^2 a = 1 - \left[\frac{2\left(\frac{\mathrm{d}\phi}{a\mathrm{d}\eta}\right)^2}{3H^2} + \Omega_k + 5\Omega_{\mathfrak{h}}\right] = -\left[1 + 2\Omega_k + 6\Omega_{\mathfrak{h}} - \sqrt{3(1 + 2\Omega_k + 2\Omega_{\mathfrak{h}})}\,\right], \tag{3.23}$$

which can be readily shown to assume negative values, particularly when $\Omega_k + \Omega_{\mathfrak{h}} > 1$. Such acceleration is only achievable in string frame, where the string frame metric minimally couples to point particles (2.14), thereby ensuring that the equivalence principle remains valid and simultaneously exhibiting a negative sign in the dilaton's kinetic term (2.118). The dilaton drives the acceleration in string frame, without any need for dark energy, but not in Einstein frame.

ii) As dictated by the $\mathbf{O}(D, D)$ symmetry principle, the string dilaton is expected to couple to gauge bosons, as demonstrated in (2.152), implying that the fine-structure constant is effectively proportional to the exponential of the string dilaton:

$$\alpha_{\text{eff.}}(\eta) = \alpha\, e^{2\phi(\eta)}\,. \tag{3.24}$$

However, observational constraints from the absorption spectra of quasars impose stringent limits on the temporal variation of the fine-structure constant [109–112], and consequently on the evolution of the string dilaton ϕ. The analytic formula (3.16) shows that the string dilaton converges at future infinity, $\eta \to \infty$. Replacing l and ζ with imaginary numbers, $-il$ and $i\zeta$, yields the exact geometry of a closed Universe ($k > 0$) (see [108] for the explicit expression). However, this substitution transforms the converging hyperbolic tangent functions in (3.16) into diverging tangent functions, preventing the dilaton ϕ from stabilising during cosmic evolution, which conflicts with the observational constraints from quasars. Similarly, when $k = 0$, the hyperbolic tangent functions reduce to linear behaviour in time, which is equally inconsistent with the observational requirements. These results underscore the necessity of an open Universe ($k < 0$) to ensure that the dilaton ϕ evolves slowly and remains convergent throughout the cosmic evolution.

iii) The vacuum geometry of the open Universe given in (3.16) and (3.19), in particular the case with trivial H-flux ($\mathfrak{h} = 0$ in (3.17)), demonstrates remarkable agreement with late-time cosmological observations, including type Ia supernovae data [113, 114] and quasar absorption spectra [109–112], as depicted in Figure 1, reproduced from [93]. Such observations probe the evolution of the Hubble parameter and the potentially varying fine-structure constant up to redshift $z = a^{-1} - 1 \approx 7$.

Through an analysis of Bayesian inference, the Hubble constant and the curvature density parameter are estimated to be

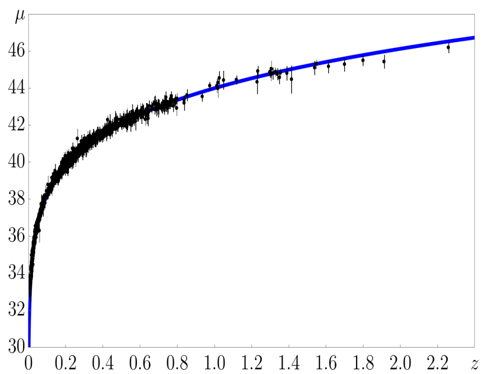
$$H_0 \simeq 71.29 \pm 0.12\,\mathrm{km/s/Mpc}\,, \qquad \Omega_{0,k} \simeq 1 + (6 \pm 2) \times 10^{-7}\,. \tag{3.25}$$

These imply a curvature scale of $l = 1/\sqrt{-k} \simeq 4.2\,\mathrm{Gpc}$ and select the negative sign factor, $\sigma = -1$, in (3.22). Such results are consistent with the analytic limiting behaviours of the scale factor (3.19):

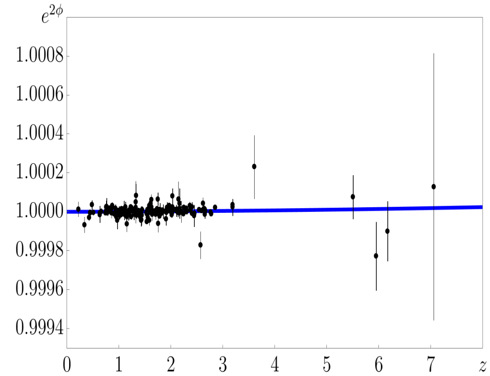
$$\lim_{\eta\to\infty} a = \infty\,, \qquad \lim_{\eta\to\infty} H = 0\,, \qquad \lim_{\eta\to\infty} \Omega_k = 1\,, \qquad \lim_{\eta\to\infty} \Omega_{\mathfrak{h}} = 0\,. \tag{3.26}$$

In other words, there is no coincidence problem.

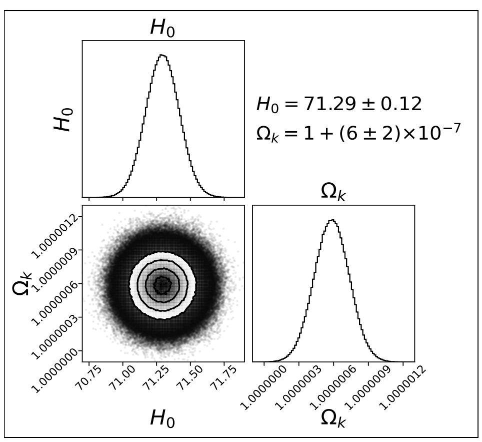
iv) Extending to higher redshifts, a bounce is expected to have occurred approximately 13.7 gigayears ago. Coincidentally, this is comparable to the "age" of the flat Universe in ΛCDM.



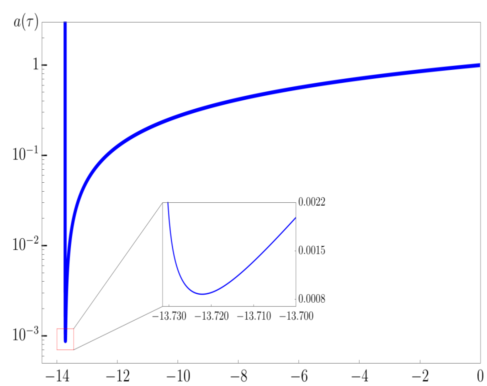
(a) Type Ia Supernovae [113, 114]



(b) Quasar Absorption Spectra [109–112]



(c) Bayesian Inference with Two Parameters



(d) Bouncing: 13.7 Gigayears ago

Figure 1: Agreement of the vacuum geometry of DFT (3.16), (3.19) with late-time cosmological data of type Ia supernovae (a) and quasars (b). For the latter, the string dilaton ϕ already appears to be in the convergent phase. Bayesian inference (c) shows that, in sharp contrast to ΛCDM, at present time the curvature density overwhelmingly dominates, $\Omega_{0,k} \simeq 1 + (6 \pm 2) \times 10^{-7}$, implying an open Universe. A bounce (d) is extrapolated to occur 13.7 gigayears ago. Figures reproduced from [93].

Summary. An open Universe with negative curvature replaces de Sitter space. The dilaton drives late-time acceleration in string frame without dark energy, the solution fits supernova and quasar observations, and it predicts a cosmological bounce about 13.7 Gyr ago.

4 Concluding Remarks & Outlook

Double Field Theory (DFT) has developed into a self-sustaining autonomous theory of gravity, offering a compelling alternative to General Relativity (GR). Governed by the $\mathbf{O}(D,D)$ symmetry principle rooted in string theory, DFT uniquely fixes the self-interaction of the gravitational sector, $\{\mathcal{H}_{AB},d\}$ or the Riemannian trio $\{g,B,\phi\}$, as well as its minimal coupling to other sectors or additional matter, placing it in a distinct position among various modified theories of gravity. The rigorously defined structure of DFT makes it inherently falsifiable, a critical feature for endowing theoretical physics with value.

The spherically symmetric solution (3.1) has been evaluated against solar system experiments using the Parametrised Post-Newtonian (PPN) formalism. Unlike General Relativity (GR), where Birkhoff's theorem holds and a single component of the energy-momentum tensor determines the geometry, Double Field Theory incorporates multiple components (3.13), enabling the exploration of more complex phenomena. For example, studying the geometry outside compact objects or stars may offer insights into their generalised equation of state (3.15) or the generalised gravitational form factor within hadrons, expressed in terms of $K_{\mu\nu}$ and $T_{(0)}$ rather than the traditional $T_{\mu\nu}$. Notably, recent experimental findings, such as the discovery of immense pressure inside protons [115], challenge traditional assumptions of matter being "cold" and may underscore the relevance of DFT's multi-component energy-momentum tensors. While these possibilities are intriguing, solar system tests of DFT remain inconclusive, necessitating further investigations, including insights from nuclear physics.

On cosmological scales, the open-Universe solution given through (3.16—3.19) captures accelerating expansion and demonstrates strong consistency with late-time cosmological observations, including type Ia supernovae and quasars. However, the theory's applicability to the early Universe remains uncertain. The non-convergent phase of the string dilaton during the early Universe suggests a nontrivial coupling to electromagnetism (2.152), (3.24), which may influence light propagation or null geodesics, thereby affecting measurements of luminosity distance. Further corroboration, incorporating diverse data sources such as the cosmic microwave background [116], is essential to clarify DFT's role in explaining the early Universe.

In terms of symmetry, DFT distinguishes itself by being approximately twice as symmetrical as General Relativity. This profound structure invites speculation about whether Nature and the Universe might harness such symmetry. As a falsifiable prediction of string theory, DFT stands as a promising candidate awaiting experimental verification.

Beyond its phenomenological significance, DFT, as the gravitational framework of string theory, faces the critical challenge of addressing extra dimensions and higher-derivative corrections. A compelling and still largely unexplored possibility regarding the former is whether these extra dimensions could take a non-Riemannian form [44, 117]. Regarding the latter, some pioneering contributions include [118–121].

Lastly, the quantisation of Double Field Theory in an $\mathbf{O}(D,D)$-symmetric framework remains a vital and promising avenue for further investigation.

While the aforementioned topics represent potential future works within the limitations of the author's imagination, the over 100-year legacy of GR suggests that DFT could unlock far broader and richer gravitational-related research directions. The development of DFT is an ongoing process, with immense potential to deepen our understanding of gravity.

Acknowledgements

The author extends sincere gratitude to Manu Paranjape, Nobuyoshi Ohta, and an anonymous peer in Kyoto for their encouraging suggestions regarding the writing of these lecture notes, and to Stephen Angus for his meticulous proofreading and invaluable feedback. This work is supported by the National Research Foundation of Korea (NRF) through the Grants RS-2023-NR077094 and RS-2020-NR049598 (Center for Quantum Spacetime: CQUeST).

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